Master of Science (Mathematics) (DDE)

Semester – II

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INTEGRAL EQUATIONS AND CALCULUS OF VARIATIONS

DIRECTORATE OF DISTANCE EDUCATION

MAHARSHI DAYANAND UNIVERSITY, ROHTAK

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Integral Equations and Calculus of Variations

M. Marks = 100

Term End Examination = 80

Time = 3 Hours ***Assignment = 20***

Course Outcomes

Students would be able to:

CO1 Understand the methods to reduce Initial value problems associated with linear differential equations to various integral equations.

CO2 Categorise and solve different integral equations using various techniques.

CO3 Describe importance of Green's function method for solving boundary value problems associated with non-homogeneous ordinary and partial differential equations, especially the Sturm-Liouville boundary value problems.

CO4 Learn methods to solve various mathematical and physical problems using variational techniques.

Section - I

Linear Integral equations, Some basic identities, Initial value problems reduced to Volterra integral equations, Methods of successive substitution and successive approximation to solve Volterra integral equations of second kind, Iterated kernels and Neumann series for Volterra equations. Resolvent kernel as a series. Laplace transfrom method for a difference kernel. Solution of a Volterra integral equation of the first kind.

Section - II

Boundary value problems reduced to Fredholm integral equations, Methods of successive approximation and successive substitution to solve Fredholm equations of second kind, Iterated kernels and Neumann series for Fredholm equations. Resolvent kernel as a sum of series. Fredholmresolvent kernel as a ratio of two series. Fredholm equations with separable kernels. Approximation of a kernel by a separable kernel, Fredholm Alternative, Non homonogenous Fredholm equations with degenerate kernels.

Section - III

Green function, Use of method of variation of parameters to construct the Green function for a nonhomogeneous linear second order boundary value problem, Basic four properties of the Green function, Alternate procedure for construction of the Green function by using its basic four properties. Reduction of a boundary value problem to a Fredholm integral equation with kernel as Green function, Hilbert-Schmidt theory for symmetric kernels.

Section - IV

Motivating problems of calculus of variations, Shortest distance, Minimum surface of resolution, Brachistochrone problem, Isoperimetric problem, Geodesic. Fundamental lemma of calculus of variations, Euler equation for one dependant function and its generalization to 'n' dependant functions and to higher order derivatives. Conditional extremum under geometric constraints and under integral constraints.

Note : The question paper of each course will consist of **five** Sections. Each of the sections **I to IV** will contain **two** questions and the students shall be asked to attempt **one** question from each. **Section-V** shall be **compulsory** and will contain **eight** short answer type questions without any internal choice covering the entire syllabus.

Books Recommended:

1. Jerri, A.J., Introduction to Integral Equations with Applications, A Wiley-Interscience Publication, 1999.
2. Kanwal, R.P., Linear Integral Equations, Theory and Techniques, Academic Press, New York.
3. Lovitt, W.V., Linear Integral Equations, McGraw Hill, New York.
4. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.
5. Gelfand, J.M., Fomin, S.V., Calculus of Variations, Prentice Hall, New Jersey, 1963.

Contents

1

Volterra Integral Equations

Structure

1.1. Introduction. This chapter contains basic definitions and identities for integral equations, various methods to solve Volterra integral equations of first and second kind. Iterated kernels and Neumann series for Volterra equations.

1.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

i. Initial value problem reduced to Volterra integral equations.

ii. Method of successive substitution to solve Volterra integral equation of second kind.

iii. Method of successive approximation to solve Volterra integral equation of second kind.

iv. Resolved kernel as a series.

v. Laplace transform method for a difference kernel.

1.1.2. Keywords. Integral Equations, Volterra Integral Equations, Iterated Kernels.

1.2. Integral Equation. An integral equation is one in which function to be determined appears under the integral sign. The most general form of a linear integral equation is

$$h(x)\, u(x) = f(x) + \int_a^{b(x)} K(x\,,\ \xi)\, u(\xi)\, d\xi \quad \text{for all } x \in [a, b]$$

in which, u(x) is the function to be determined and **K(x, ξ) is called the Kernel of integral equation**.

1.2.1. Volterra Integral equation. A Volterra integral equation is of the type:

$$h(x)\, u(x) = f(x) + \int_a^{x} K(x\,,\ \xi)\, u(\xi)\, d\xi \quad \text{for all } x \in [a, b]$$

that is, in Volterra equation b(x) = x

(i) If h(x) = 0, the above equation reduces to

$$-\, f(x) = \int_a^{x} K(x\,,\ \xi)\, u(\xi)\, d\xi$$

This equation is called **Volterra integral equation of first kind**.

(ii) If h(x) = 1, the above equation reduces to

$$u(x) = f(x) + \int_a^{x} K(x\,,\ \xi)\, u(\xi)\, d\xi$$

This equation is called **Volterra integral equation of second kind**.

1.2.2. Homogeneous integral equation. If f(x) = 0 for all x $\in$ [a, b], then the reduced equation

$$h(x)\, u(x) = \int_a^{bx} K(x\,,\ \xi)\, u(\xi)\, d\xi$$

is called homogeneous integral equation. Otherwise, it is called non-homogeneous integral equation.

1.2.3. Leibnitz Rule. The Leibnitz rule for differentiation under integral sign:

$$\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} F(x\,,\ \xi)\, d\xi\right] = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x}\, d\xi + F(x\,,\ \beta(x))\, \frac{d\beta(x)}{dx} - F(x\,,\ \alpha(x))\, \frac{d\alpha(x)}{dx}$$

In particular, we have

$$\frac{d}{dx}\left[\int_a^{x} K(x\,,\ \xi) u(\xi)\, d\xi\right] = \int_a^{x} \frac{\partial K}{\partial x} u(\xi)\, d\xi + K(x\,,\ x)\, u(x)\,.$$

1.2.4. Lemma. If n is a positive integer, then

$$\int_a^x\int_a^{x_1}\cdots\int_a^{x_{n-2}}\int_a^{x_{n-1}} F(x_n)\,dx_n\,dx_{n-1}\ldots dx_1 = \frac{1}{n-1!}\int_a^x (x-\xi)^{n-1} f(\xi)\,d\xi\,.$$

Proof. If $I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)\,d\xi$, then $I_n(a) = 0$ and for n = 1, $I_1(x) = \int_a^x f(\xi)\,d\xi$.

Using Leibnitz rule, we get $\frac{dI_1}{dx} = f(x)$.

Now, differentiating $I_n(x)$ w.r.t. x and using Leibnitz rule, we get

$$\frac{dI_1}{dx} = \frac{d}{dx}\,.\ \ . = \int_a^x \frac{\partial}{\partial x}\left[(x-\xi)^{n-1}\right] f(\xi)\,d\xi = (n-1)\int_a^x (x-\xi)^{n-2} f(\xi)\,d\xi$$

or $$\frac{dI_n(x)}{dx} = (n-1)\ I_{n-1}(x) \text{ for } n > 1$$

Taking successive derivatives, we get

$$\frac{d^{n-1}}{dx^{n-1}}\ I_n(x) = (n-1)\,(n-2)\ldots 2.1\ I_1(x)$$

Again, differentiating,

$$\frac{d^n}{dx^n} I_n(x) = n-1!\ \frac{d}{dx}\ I_{1(x)} = n-1!\ f(x) \qquad (1)$$

We observe that,

$$I_n^{(m)}(a) = 0 \text{ for } m = 0, 1, 2, \ldots, n-1 \qquad (2)$$

Integrating (1) over the interval [a, x] and using (2) for m = n − 1, we obtain

$$I_n^{(n-1)}(x) = (n-1)!\int_a^x f(x_1)\,dx_1$$

Again integrating it and using (2) for m = n − 2, we get

$$\frac{d^{n-2}}{dx^{n-2}} I_n(x) = I_n^{(n-2)}(x) = n-1!\int_a^x\int_a^{x_1} f(x_2)\,dx_2\,dx_1$$

Continuing like this, n times, we obtain

$$I_n(x) = (n-1)!\int_a^x\int_a^{x_1}\ldots\ldots\int_a^{x_{n-1}} f(x_n)\,dx_n\,dx_{n-1}\ldots\ldots dx_1\,.$$

which provides the required result.

1.2.5. Example. Transform the initial value equation $\frac{d^2y}{dx^2}$ + x $\frac{dy}{dx}$ +y = 0; y(0) = 1, $y'(0) = 0$ to Volterra integral equation.

Solution. Consider,
$$\frac{d^2y}{dx^2} = \phi(x) \qquad (1)$$

Then
$$\frac{dy}{dx} = \int_0^x \phi(\xi)\, d\xi + C_1$$

Using the condition $y'(0) = 0$, we get, $C_1 = 0$

$$\frac{dy}{dx} = \int_0^x \phi(\xi)\, d\xi \qquad (2)$$

Again, integrating from 0 to x and using the above lemma, we get

$$y = \int_0^x (x-\xi)\, \phi(\xi)\, d\xi + C_2$$

Using the condition y(0) = 1, we get $C_2 = 1$

So,
$$y = \int_0^x (x-\xi)\, \phi(\xi)\, d\xi + 1 \qquad (3)$$

From the relations (1), (2) and (3), the given differential equation reduces to :

$$\phi(x) + x\int_0^x \phi(\xi)\, d\xi + \int_0^x (x-\xi)\, \phi(\xi)\, d\xi + 1 = 0$$

or
$$\phi(x) = -1 - \int_0^x (2x-\xi)^{n-1}\phi(\xi)\, d\xi$$

which represents a Volterra integral equation of second kind.

1.2.6. Exercise. Reduce following initial value problem into Volterra integral equations:

1. $y'' + xy = 1$, $y'(0) = 0 = y(0)$.

Answer. $y(x) = \frac{x^2}{2} - \int_0^x (x-\xi)\, \xi\, y(\xi)\, d\xi$.

2. $\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = g(x)$, $y(a) = c_1$ and $y'(a) = c_2$.

Answer. $f(x) = c_1 + c_2(x-a) + \int_a^x (x-\xi)\, g(\xi)\, d\xi + A(a)\, c_1(x-a)$,

where $K(x, \xi) = (x-\xi)\left[A'(\xi) - B(\xi)\right] - A(\xi)$.

3. $y'' + \lambda y = 0$, $y(0) = 1$, $y'(0) = 0$.

Answer. $y(x) = 1 - \lambda \int_0^x (x-\xi)\; y(\xi)\, d\xi$.

4. $y'' - 5y' + 6y = 0$, $y(0) = 0$, $y'(0) = -1$.

Answer. $y(x) = (6x-5) + \int_0^x (5-6x+6\xi)\, \phi(\xi)\, d\xi$.

1.3. Solution of Volterra Integral Equation.

1.3.1. Weierstrass M-Test. Suppose $\sum f_n(z)$ is an infinite series of single valued functions defined in a bounded closed domain D. Let $\sum M_n$ be a series of positive constants (independent of z) such that

(i) $|f_n(z)| \le M_n$ for all n and for all $z \in D$.

(ii) $\sum M_n$ is convergent.

Then the series $\sum f_n$ is uniformly and absolutely convergent in D.

1.3.2. Theorem. Let $u(x) = f(x) + \lambda \int_a^x K(x, \xi)\, u(\xi)\, d\xi$ be a non-homogeneous Volterra integral equation of second kind with constants a and λ. f(x) is a non-zero real valued continuous function in the interval I = [a, b]. $K(x, \xi)$ is a non-zero real valued continuous function defined in the rectangle $R = I \times I = \{(x, \xi) : a \le x, \xi \le b\}$ and $|K(x, \xi)| \le M$ in R.

Then the given equation has one and only one continuous solution u(x) in I and this solution is given by the absolutely and uniformly convergent series.

$$u(x) = f(x) + \lambda \int_a^x K(x, t)\, f(t)\, dt + \lambda^2 \int_a^x \int_a^t K(x, t)\, K(t, t_1)\, f(t_1)\, dt_1 dt + \ldots$$

Proof. This theorem can be proved by applying either of the following two methods :

(a) Method of successive substitution.

(b) Method of successive approximation.

Let us apply these methods one by one.

(a) Method of Successive Substitution. The given integral equation is

$$u(x) = f(x) + \lambda \int_a^x K(x,\ t)\ u(t)\ dt \tag{1}$$

Substituting value of u(t) from (1) into itself, we get

$$u(x) = f(x) + \lambda \,.\int_a^x K(x,\ t)\left[\ f(t) + \lambda \int_a^t K(t,\ t_1)\ u(t_1)\ dt_1\right] dt\ .$$

$$= f(x)\ \ \lambda \int_a^x K(x,\ t)\ f(t)\ dt + \lambda^2 \int_a^x\int_a^t K(x,\ t)\ K(t,\ t_1)\ u(t_1)\ dt_1 dt \tag{2}$$

Again substituting the value of $u(t_1)$ from (1) into (2), we get

$$u(x) = f(x) + \lambda \int_a^x K(x,\ t)\ f(t)\ dt + \lambda^2 \int_a^x\int_a^t K(x,\ t)\ K(t,\ t_1)\ f(t_1)\ dt_1 dt$$

$$+ \lambda^3 \int_a^x\int_a^t\int_a^{t_1} K(x,\ t)\ K(t,\ t_1) K(t_1,\ t_2)\ u(t_2)\ dt_2 dt_1 dt$$

Proceeding in the same way, we get after n steps

$$\begin{aligned} u(x) &= f(x) + \lambda \int_a^x K(x\ ,\ t)\ f(t) dt \\ &+ \ldots + \lambda^n \int_a^x\int_a^t \ldots \int_a^{t_{n-2}} K(x,\ t)\ K(t,\ t_1) \ldots K(t_{n-2}, t_{n-1})\ f(t_{n-1}) dt_{n-1}\ dt_{n-2} \ldots . dt_1 dt + R_{n+1}(x) \end{aligned} \tag{3}$$

where $R_{n+1}(x) = \lambda^{n+1} \int_a^x\int_a^t \ldots \int_a^{t_{n-1}} K(x,\ t)\ K(t,\ t_1) \ldots K(t_{n-1}, t_n)\ u(t_n) dt_n dt_{n-1} \ldots dt_1 dt$ (4)

Consider the infinite series,

$$f(x) + \lambda \int_a^x K(x,\ t)\ f(t) dt\ + \lambda^2 \int_a^x\int_a^t K(x,\ t)\ K(t,\ t_1)\ f(t_1)\ dt_1 dt + \ldots \tag{5}$$

Neglecting the first term, let $v_n(x)$ denotes the nth term of infinite series in (5). Since f(x) is continuous over I, so it is bounded.

Let $|f(x)| \le N$ in I. Also, it is given that $|K(x,\ t)| \le M$ in R. Therefore,

$$|v_n(x)| \le |\lambda|^n \int_a^x \int_a^t \ldots \int_a^{t_{n-2}} M^n \, N \, dt_{n-1} \ldots dt_1 \, dt$$

Thus, $$|v_n(x)| \le |\lambda|^n M^n N \frac{(x-a)^n}{n!} \le |\lambda|^n M^n N \frac{(b-a)^n}{n!} \quad (6)$$

The series whose nth term is $|\lambda|^n M^n N \frac{(b-a)^n}{n!}$ is a series of positive terms and is convergent by ratio test for all values of a, b, $|\lambda|$, M and N.

Thus, by Weierstrass M–test, the series $\sum v_n(x)$ is absolutely and uniformly convergent in I .

If u(x) given by (2) is continuous in I, then is bounded in I, that is,

u(x) ≤ U for all x in I (7)

Then, $|R_{n+1}(x)| \le |\lambda|^{n+1} M^{n+1} u \frac{(x-a)^{n+1}}{(n+1)!} \le |\lambda|^{n+1} M^{n+1} u \frac{(b-a)^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$

$$\Rightarrow \qquad \lim_{n \to \infty} R_{n+1}(x) = 0 \quad (8)$$

From equations (3), (4) and (8), we obtain

$$u(x) = f(x) + \lambda \int_a^x K(x, t) \, f(t) \, dt + \lambda^2 \int_a^x \int_a^t K(x, t) \, K(t, t_1) \, f(t_1) \, dt_1 \, dt + \ldots \text{to } \infty$$

which is the required series.

Now, we verify that this series is actually a solution of the given Volterra integral (1). Substituting the series for u(x) in the R.H.S. of the given equation, we get

$$\text{R.H.S.} = f(x) + \lambda \int_a^x K(x, \xi) \left[f(\xi) + \lambda \int_a^{\xi} K(\xi, t) \, f(t) \, dt + \lambda^2 \int_a^{\xi} \int_a^t K(\xi, t) \, K(t, t_1) \, f(t_1) \, dt_1 dt + \ldots \text{to } \infty \right]$$

$$= f(x) + \lambda \int_a^x K(x, \xi) \, f(\xi) \, d\xi + \lambda^2 \int_a^x \int_a^{\xi} K(x, \xi) \, K(\xi, t) \, f(t) \, dt \, d\xi + \ldots \text{to } \infty = u(x) = \text{L.H.S.}$$

(b) Method of Successive Approximation. In this method, we select any real valued function, say $u_0(x)$, continuous on I = [a, b] as the zeroth approximation. Substituting this zeroth approximation in the given Volterra integral equation.

$$u(x) = f(x) + \lambda \int_a^x K(x, t) \, u(t) \, dt \quad (1)$$

We obtain the first approximation, say $u_1(x)$, given by

$$u_1(x) = f(x) + \lambda \int_a^x K(x\,,\,t)\, u_0(t)\, dt \tag{2}$$

The value of $u_1(x)$ is again substituted for $u(x)$ in (1) to obtain the second approximation, $u_2(x)$ where

$$u_2(x) = f(x) + \lambda \int_a^x K(x\,,\,t)\, u_1(t)\, dt \tag{3}$$

This process is continued to obtain n^{th} approximation

$$u_n(x) = f(x) + \lambda \int_a^x K(x\,,\,t)\, u_{n-1}(t)\, dt \quad \text{for } n = 1, 2, 3, \ldots \tag{4}$$

This relation is known as recurrence relation.

Now, we can write

$$u_n(x) = f(x) + \lambda \int_a^x K(x,\, t) \left[f(t) + \lambda \int_a^t K(t,\, t_1)\, u_{n-2}(t_1)\, dt_1 \right] dt$$

$$= f(x) + \lambda \int_a^x K(x,\, t)\, f(t)\, dt + \lambda^2 \int_a^x \int_a^t K(x,\, t)\, K(t,\, t_1) \left[f(t_1) + \lambda \int_a^{t_1} K(t_1,\, t_2) u_{n-3}(t_2)\, dt_2 \right] dt_1\, dt$$

or $$u_n(x) = f(x) + \lambda \int_a^x K(x,\, t)\, f(t)\, dt + \lambda^2 \int_a^x \int_a^t K(x,\, t)\, K(t,\, t_1) f(t_1) dt_1 dt$$

$$+ \lambda^3 \int_a^x \int_a^t \int_a^{t_1} K(x,\, t)\, K(t,\, t_1)\, K(t_1,\, t_2) dt_2\, dt_1 dt \tag{5}$$

Continuing in this fashion, we get

$$u_n(x) = f(x) + \lambda \int_a^x K(x,\, t)\, f(t)\, dt + \ldots$$

$$+ \lambda^{n-1} \int_a^x \int_a^t \ldots \int_a^{t_{n-3}} K(x,\, t)\, K(t,\, t_1) \ldots K(t_{n-3},\, t_{n-2})\, f(t_{n-2}) dt_{n-2} \ldots dt_1 dt + R_n(x)\mathbb{F} \tag{6}$$

where

$$R_n(x) = \lambda^n \int_a^x \int_a^t \ldots \int_a^{t_{n-2}} K(x,\ t)\ K(t,\ t_1)\ldots K(t_{n-2},\ t_{n-1})\, u_0\,(t_{n-1}) dt_{n-1}\ldots dt_1 dt \qquad (7)$$

Since $u_0(x)$ is continuous on I so it is bounded.

Let $$|u_0(x)| \ \le\ u \ \text{ in } \ I \qquad (8)$$

Thus, $|R_n(x)| \le |\lambda|^n M^n u \dfrac{(x-a)^n}{n!} \le |\lambda|^n M^n u \dfrac{(b-a)^n}{n!} \to 0 \text{ as } n \to \infty$

So, $$\lim_{n \to \infty} R_n(x) = 0. \qquad (9)$$

Thus, as n increases, the sequence < $u_n(x)$ > approaches to a limit. We denote this limit by u(x) that is,

$$u(x) = \lim_{n\to\infty} u_n(x)$$

So, $$u(x) = f(x) + \lambda \int_a^x K(x,\ t)\ f(t)\ dt\ + \lambda^2 \int_a^x \int_a^t K(x,\ t)\ K(t,\ t_1)\ f(t_1)\ dt_1 dt + \ldots \text{to } \infty \qquad (10)$$

As in the method of successive substitution, we can prove that the series (10) is absolutely and uniformly convergent and hence the series on R.H.S. of (10) is the desired solution of given Volterra integral equation.

Uniqueness. Let, if possible, the given Volterra integral equation has another solution v(x). We make, by our choice, the zeroth approximation $u_0(x) = v(x)$, then all approximations $u_1(x),\ldots,u_n(x)$ will be identical with v(x) that is,

$$u_n(x) = v(x) \text{ for all n}$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n(x) = v(x)$$

$$\Rightarrow \quad u(x) = v(x)$$

This proves uniqueness of solution. With this, the proof of the theorem is completed.

1.3.3. Example. Using the method of successive approximation solve the integral equation,

$$u(x) = x - \int_0^x (x-\xi)\ u(\xi)\ d\xi \qquad (1)$$

Solution. Let the zeroth approximation be $u_0(x) = 0$

Then the first approximation $u_1(x)$ is given by :

$$u_1(x) = x - \int_0^x 0.\ d\xi\ = x \qquad (2)$$

Thus, $u_2(x) = x - \int_0^x (x-\xi)\, u_1(\xi)\, d\xi = x - \int_0^x (x-\xi)\, \xi\, d\xi$

$$= x - \left[\frac{x\xi^2}{2}\right]_0^x + \left[\frac{\xi^3}{3}\right]_0^x = x - \frac{x^3}{2} + \frac{x^3}{3}$$

$$= x - \frac{x^3}{6} = x - \frac{x^3}{3!} \tag{3}$$

Now, $u_3(x) = x - \int_0^x (x-\xi)\, u_2(\xi)\, d\xi$

$$= x - \int_0^x (x-\xi) \left(\xi - \frac{\xi^3}{6}\right) d\xi$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{4}$$

From (2), (3) and (4), we conclude that the nth approximation, $u_n(x)$ will be

$$u_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \tag{5}$$

which is obviously the nth partial sum of Maclaurin's series of sinx. Hence by the method of successive approximation, solution of given integral equation is

$$u(x) = \lim_{n\to\infty} u_n(x) = \sin x$$

Hence the solution.

1.3.4. Exercise.

1. Using the method of successive approximation, solve the integral equation,

$$y(x) = e^x + \int_0^x e^{x-t}\, y(t)\, dt .$$

Answer. $y(x) = \lim_{n\to\infty} e^x \left[1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}\right] = e^x . e^x = e^{2x}$.

2. $u(x) = 1 + \int_0^x (x-\xi)\, u(\xi)\, d\xi$.

Answer. cosh x

3. $u(x) = 1 + \int_0^x (\xi - x)\, u(\xi)\, d\xi$

Answer. cos x

4. $u(x) = 1 + \int_0^x u(\xi)\, d\xi$

Answer. e^x

5. $u(x) = e^{x^2} + \int_0^x e^{x^2 - t^2}\, u(t)\, dt$

Answer. $e^{x\,(x+1)}$

6. $u(x) = (1 + x) + \int_0^x (x - \xi)\, u(\xi)\, d\xi$ with $u_0(x) = 1$

Answer. e^x

1.4. Laplace transform method to solve an integral equation.

1.4.1. Definition. The Laplace transform of a function f(x) defined on interval $(0, \infty)$ is given by

$$L[f(x)] = f(s) = \int_0^\infty f\,(x)\, e^{-sx} dx \qquad (1)$$

Here, s is called Laplace variable or Laplace parameter. Also . $f(x) = L^{-1}[f(s)]$ is called inverse Laplace transform.

1.4.2. Some important results.

(1) $L(\sin x) = \frac{1}{s^2+1}$ (2) $L[\cos x] = \frac{s}{s^2+1}$ (3) $L[e^{ax}] = \frac{1}{s-a}$

(4) $L[x^n] = \frac{n!}{s^n+1}$, $n \geq 0$ (5) $L[f'(x)] = s\,f(s) - f(0)$ (6) $L[1] = \frac{1}{s}$.

1.4.3. Convolution. The convolution of two functions $f_1(x)$ and $f_2(x)$ is denoted by

$(f_1 * f_2)(x)$ and is defined as $(f_1 * f_2)(x) = \int_0^x f_1(x-\xi)\, f_2(\xi)\, d\xi$

1.4.4. Convolution theorem.(without proof)

Laplace transform of convolution of two functions is equal to the product of their respective Laplace transforms, that is, $\left[(f_1 * f_2)(x)\right] = L[f_1(x)]\,.\,L[f_2(x)]$.

1.4.5. Difference Integral or Convolution Integral. Consider the integral equation

$$u(x) = f(x) + \int_a^{b(x)} K(x,\xi)\, u(\xi)\, d\xi$$

Let the kernel K(x, ξ) be a function of x $-$ ξ, say g(x $-$ ξ) then the integral equation becomes

$$u(x) = f(x) + \int_a^{b(x)} g(x-\xi)\, u(\xi)\, d\xi$$

In this case, the kernel **K(x, ξ) = g(x − ξ) is called difference kernel** and the corresponding integral is called difference integral or convolution integral.

1.4.6. Working Procedure. Consider the integral equation

$$u(x) = f(x) + \lambda \int_0^{x} K(x,\xi)\, u(\xi)\, d\xi$$

where K(x, ξ) is difference kernel of the type g(x $-$ ξ) then,

$$u(x) = f(x) + \lambda \int_0^{x} g(x-\xi)\, u(\xi)\, d\xi$$

$$\Rightarrow \qquad u(x) = f(x) + \lambda\left[g(x) * u(x)\right]$$

Applying Laplace transform on both sides, we get

$$U(s) = F(s) + \lambda\, G(s)\, U(s)$$

where U(s), F(s) and G(s) represent the Laplace Transform of u(x), f(x) and g(x) respectively.

Then, $$U(s) = \frac{F(s)}{1-\lambda G(s)}$$

Applying inverse Laplace Transform

$$u(x) = L^{-1}\left[\frac{F(s)}{1-\lambda G(s)}\right]$$

Note. Method of Laplace Transform is applicable to those integral equations only where the kernel is difference Kernel.

1.4.7. Example. Use the method of Laplace Transform to solve the integral equation.

$$u(x) = x - \int_0^{x} (x-\xi)\, u(\xi)\, d\xi \qquad (1)$$

Solution. Here $K(x, \xi) = x - \xi = g(x-\xi) \Rightarrow g(x) = x$

Thus, (1) can be written as $u(x) = x - g(x) * u(x)$

Applying Laplace Transform on both sides

$$U(s) = L[x] - L[x]\, U(s)$$

$$= \frac{1}{s^2} - \frac{1}{s^2} U(s)$$

$$\Rightarrow \qquad U(s) = \frac{\frac{1}{s^2}}{1+\frac{1}{s^2}} = \frac{1}{s^2+1}$$

So, $$u(x) = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin x.$$

1.4.8. Exercise. Use the method of Laplace Transform to solve the following integral equations.

(1) $u(x) = 1 + \int_0^x (x-\xi)\, u(\xi)\, d\xi$

Answer. cosh x.

(2) $u(x) = 1 + \int_0^x (\xi - x)\, u(\xi)\, d\xi$

Answer. cos x.

(3) $u(x) = 1 + \int_0^x u(\xi)\, d\xi$

Answer. e^x

1.5. Solution of Volterra Integral Equation of first kind. Consider the non-homogeneous Volterra integral equation of first kind

$$f(x) = \lambda \int_0^x K(x, \xi)\, u(\xi)\, d\xi \tag{1}$$

Where the kernel $K(x, \xi)$ is the difference Kernel of the type

$$K(x, \xi) = g(x-\xi)$$

Then (1) can be written as

$$f(x) = \lambda\, g(x) * u(x)$$

Applying Laplace Transform on both sides :

$$F(s) = \lambda\ G(s)\ U(s)$$

$$\Rightarrow \qquad U(s) = \frac{1}{\lambda}\ \frac{F(s)}{G(s)}$$

Applying inverse Laplace Transform on both sides :

$$u(x) = \frac{1}{\lambda}\ L^{-1}\left[\frac{F(s)}{G(s)}\right]$$

1.5.1. Example. Solve the integral equation $\sin x = \lambda \int_0^x e^{x-\xi}\ u(\xi)\ d\xi$ (1)

Solution. Here $K(x, \xi) = e^{x-\xi} = g(x-\xi)$

$$\Rightarrow \qquad g(x) = e^x$$

Thus, (1) can be written as

$$\sin x = \lambda\ g(x) * u(x)$$

Applying Laplace Transform on both sides

$$L[\sin x] = \lambda\ L[e^x]\ L[u(x)]$$

$$\Rightarrow \qquad \frac{1}{s^2+1} = \frac{\lambda}{s-1}\ U(s)$$

$$\Rightarrow \qquad U(s) = \frac{1}{\lambda}\ \frac{s-1}{s^2+1} = \frac{1}{\lambda}\left[\frac{s}{s^2+1} - \frac{1}{s^2+1}\right]$$

So, $$u(x) = \frac{1}{\lambda}\ L^{-1}\left[\frac{s}{s^2+1} - \frac{1}{s^2+1}\right]$$

$$u(x) = \frac{1}{\lambda}\ (\cos x - \sin x).$$

1.5.2. Exercise. Solve the integral equation $x = \int_0^x \cos(x-\xi)\ u(\xi)\ d\xi$.

Answer. $1+\frac{x^2}{2}$.

1.5.3. Theorem. Prove that the Volterra integral equation of first kind $f(x) = \lambda \int_0^x K(x, \xi)\ u(\xi)\ d\xi$ can be transformed to a Volterra integral equation of second kind, provided that $K(x, x) \neq 0$.

Proof. The given equation is

$$f(x) = \lambda \int_0^x K(x, \xi)\, u(\xi)\, d\xi \qquad (1)$$

Differentiating (1), w.r.t. x and using Leibnitz rule, we obtain

$$\frac{df}{dx} = \lambda \int_0^x \frac{\partial K}{\partial x} u(\xi)\, d\xi + \lambda K(x, x)\, u(x).1$$

$$\Rightarrow \qquad -\lambda K(x, x)\, u(x) = \lambda \int_0^x \frac{\partial K}{\partial x} u(\xi)\, d\xi - \frac{df}{dx}$$

$$\Rightarrow \qquad u(x) = \frac{1}{\lambda K(x, x)} . \frac{df}{dx} + \int_0^x -\frac{1}{K(x, x)} \frac{\partial K}{\partial x} u(\xi)\, d\xi$$

$$\Rightarrow \qquad u(x) = g(x) + \int_0^x H(x, \xi)\, u(\xi)\, d\xi \qquad (*)$$

where $g(x) = \dfrac{1}{\lambda K(x, x)} \dfrac{df}{dx}$ and $H(x, \xi) = \dfrac{-1}{K(x, x)} \dfrac{\partial K}{\partial x}$. Here, (*) represents the desired Volterra integral equation of second kind.

1.5.4. Example. Reduce the integral equation $\sin x = \lambda \int_0^x e^{x-\xi}\, u(\xi)\, d\xi$ to the second kind and hence solve it.

Solution. The given equation is

$$\sin x = \lambda \int_0^x e^{x-\xi}\, u(\xi)\, d\xi \qquad (1)$$

Differentiating (1) w.r.t. *x*, we get

$$\cos x = \lambda \int_0^x e^{x-\xi}\, u(\xi)\, d\xi + \lambda e^{x-x} u(x).1$$

$$\Rightarrow \qquad \cos x = \lambda \int_0^x e^{x-\xi}\, u(\xi)\, d\xi + \lambda\, u(x)$$

$$\Rightarrow \qquad u(x) = \frac{1}{\lambda} \cos x - \int_0^x e^{x-\xi}\, u(\xi)\, d\xi \qquad (2)$$

which is Volterra integral equation of second kind and can be simply solved by the method of Laplace Transform.

1.6. Method of Iterated kernel/Resolvent kernel to solve the Volterra integral equation.

Consider the Volterra integral equation

$$u(x) = f(x) + \lambda \int_a^x K(x, \xi)\, u(\xi)\, d\xi \tag{1}$$

We take $K_1(x, \xi) = K(x, \xi)$ (2)

and $K_{n+1}(x, \xi) = \int_\xi^x K(x, t)\, K_n(t, \xi)\, dt$; n = 1, 2, 3,… (3)

From here, we get a sequence of new kernels and these kernels are called iterated kernels.

We know that (1) has one and only one series solution given by

$$u(x) = f(x) + \lambda \int_a^x K(x, t)\, f(t)\, dt + \lambda^2 \int_a^x \int_a^t K(x, t)\, K(t, t_1)\, f(t_1)\, dt_1 dt + \text{...to } \infty \tag{4}$$

We write this series solution in the form :

$$u(x) = u_0(x) + \lambda\, u_1(x) + \lambda^2\, u_2(x) + \dots \text{ to } \infty \tag{5}$$

Then comparing (4) and (5), we have

$$u_0(x) = f(x)$$

$$u_1(x) = \int_a^x K(x, t)\, f(t)\, dt = \int_a^x K_1(x, t)\, f(t)\, dt$$

and $$u_2(x) = \int_a^x \int_a^t K(x, t)\, K(t, t_1)\, f(t_1)\, dt_1\, dt$$

By interchanging the order of integration, we have

$$u_2(x) = \int_a^x f(t_1) \left[\int_{t_1}^x K(x, t)\, K_1(t, t_1)\, dt \right] dt_1$$

$$= \int_a^x f(t_1)\, K_2(x, t_1)\, dt_1 = \int_a^x f(t)\, K_2(x, t) dt$$

Similarly, $$u_n(x) = \int_a^x f(t)\, K_n(x, t)\, dt$$

Thus, (5) becomes

$$u(x) = f(x) + \lambda \int_a^x K_1(x, t)\, f(t)\, dt + \lambda^2 \int_a^x K_2(x, t)\, f(t)\, dt + \dots \text{ to } \infty$$

$$\Rightarrow \qquad u(x) = f(x) + \lambda\int_a^x \left[K_1(x\ ,\ t)+\lambda\ K_2(x\ ,\ t)\ +\lambda^2\ K_3(x,t)+...\infty\right] f(t)\ dt$$

$$= f(x) + \lambda\int_a^x R(x\ ,\ t\ :\ \lambda)\ f(t)\ dt \qquad (6)$$

where $R(x, t : \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x\ ,\ t)$

Thus, (6) is the solution of given integral (1).

1.6.1. Neumann Series. The series $K_1 + \lambda K_2 + \lambda^2 K_3 + \ldots\ldots\ldots\ldots$ to ∞ is called the Neumann Series.

1.6.2. Resolvent Kernel. The sum of Neumann Series $R(x, t : \lambda)$ is called the Resolvent Kernel.

1.6.3. Example. With the aid of Resolvent Kernel find the solution of the integral equation

$$\phi(x) = x + \int_0^x (\xi - x)\ \phi\ (\xi)\ d\xi\ .$$

Solution. Here, $\quad K_1(x, \xi) = K(x, \xi) = \xi - x \qquad (1)$

and $\quad K_{n+1}(x, \xi) = \int_\xi^x K(x\ ,\ t)\ K_n(t\ ,\ \xi)\ dt \qquad (2)$

Putting n = 1, 2, 3,… in (2), we have,

$$K_2(x, \xi) = \int_\xi^x K(x\ ,\ t)\ K_1(t\ ,\ \xi)\ dt\ = \int_\xi^x (t-x)(\xi-t)\ dt = \frac{-1}{3!}(\xi-x)^3$$

and $$K_3(x, \xi) = \int_\xi^x K(x\ ,\ t)\ K_2(t\ ,\ \xi)\ dt\ = \int_\xi^x (t-x)\left[-\frac{1}{3!}(\xi-t)^3\right] dt\ = \frac{1}{5!}(\xi-t)^5$$

The Resolvent Kernel is defined as

$$R(x,\ \xi\ :\ \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1}\ K_n(x\ ,\ \xi)\ = \frac{\xi-x}{1!} - \frac{(\xi-x)^3}{3!} + \frac{(\xi-x)^5}{5!} \ldots\ldots\ldots \text{ to } \infty\ (\lambda\ = 1)$$

$$= \sin(\xi - x)$$

The solution of the integral equation is given by

$$\phi(x) = f(x) + \lambda\int_0^x R(x\ ,\ \xi\ :\ \lambda)\ f(\xi)\ d\xi$$

$$= x + \int_0^x \xi\ \sin(\xi - x)\ d\xi$$

$$= x + \sin x - x \text{ [Integrating by parts]}$$

$$= \sin x$$

This completes the solution.

1.6.4. Exercise. Obtaining the Resolvent Kernel, solve the following Volterra integral equation of second kind:

(1) $u(x) = f(x) + \lambda \int_0^x e^{x-\xi} u(\xi)\, d\xi$

Answer. $u(x) = f(x) + \lambda\, e^{(1+\lambda)x} \int_0^x e^{-(1+\lambda)\xi} f(\xi)\, d\xi$.

(2) $\phi(x) = 1 + \int_0^x (\xi - x)\, \phi(\xi)\, d\xi$

Answer. cos x.

(3) $\phi(x) = e^{x^2} + \int_0^x e^{x^2-\xi^2} \phi(\xi)\, d\xi$

Answer. $e^{x(x+1)}$.

1.7. Check Your Progress.

1. Reduce following initial value problem into Volterra integral equations:

$y'' - 2x\, y' - 3y = 0; \quad y(0) = 0,\ y'(0) = 0$

Answer. $y(x) = \int_0^x (x+\xi)\, y(\xi)\, d\xi$.

2. Using the method of successive approximation, solve the integral equation,

$u(x) = (1 + x) - \int_0^x u(\xi)\, d\xi$ with $u_0(x) = 1$.

Answer. 1.

3. Use the method of Laplace Transform to solve the following integral equations.

$u(x) = e^{-x} + \int_0^x \sin(x-\xi)\, u(\xi)\, d\xi$

Answer. $2e^{-x} - 1 + x$.

1.8. Summary. In this chapter, various methods like successive approximations, successive substitutions, resolvent kernel, Laplace transform are discussed to solve a Volterra integral equation. Also it is observed that a Volterra integral equation always transforms into an initial value problem.

Books Suggested:

1. Jerri, A.J., Introduction to Integral Equations with Applications, A Wiley-Interscience Publication, 1999.
2. Kanwal, R.P., Linear Integral Equations, Theory and Techniques, Academic Press, New York.
3. Lovitt, W.V., Linear Integral Equations, McGraw Hill, New York.
4. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.
5. Gelfand, J.M., Fomin, S.V., Calculus of Variations, Prentice Hall, New Jersey, 1963.

2

Fredholm Integral Equations

Structure

2.1. Introduction. This chapter contains definitions and various types for Fredholm integral equations, various methods to solve Fredholm integral equations of first and second kind. Resolvent kernels are used to solve Fredholm integral equations.

2.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Boundary value problem reduced to Fredholm integral equations.

(ii) Method of successive substitution to solve Fredholm integral equation of second kind.

(iii) Method of successive approximation to solve Volterra integral equation of second kind.

(iv) Iterated kernel and Neumann series for Fredholm equations.

2.1.2. Keywords. Fredholm Integral Equations, Successive Approximations, Iterated Kernels.

2.2. Fredholm Integral equation. A Fredholm integral equation is of the type

$$h(x)\, u(x) = f(x) + \int_a^b K(x,\ \xi)\, u(\xi)\, d\xi \quad \text{for all } x \in [a, b]$$

that is, b(x) = b in this case or we can say that in Fredholm integral equation both lower and upper limits are constant.

(i) If h(x) = 0, the above equation reduces to :

$$-\, f(x) = \int_a^b K(x,\ \xi)\, u(\xi)\, d\xi$$

This equation is called **Fredholm integral equation of first kind**.

(ii) If h(x) = 1, the above equation becomes :

$$u(x) = f(x) + \int_a^b K(x\ ,\ \xi)\, u(\xi)\, d\xi$$

This equation is called **Fredholm integral equation of second kind**.

2.2.1. Example. Reduce the boundary value problem to Fredholm equation,

$$y'' + xy = 1,\ y(0) = 0,\ y(1) = 0.$$

Solution. Given boundary value problem is

$$y'' = 1 - xy \tag{1}$$

Integrating over 0 to x,

$$y'(x) = x - \int_0^x \xi\, y(\xi)\, d\xi + c_1$$

Again integrating over 0 to x,

$$y(x) = \left[\frac{x^2}{2}\right]_0^x - \int_0^x (x-\xi)\, \xi\, y(\xi)\, d\xi + c_1 x + c_2 \tag{2}$$

where c_1 and c_2 are constants to be determined by boundary value conditions.

Using y(0) = 0 in (2), we get

$$0 = 0 - 0 + 0 + c_2 \qquad \Rightarrow \qquad c_2 = 0$$

So, (2) becomes

$$y(x) = \frac{x^2}{2} - \int_0^x (x-\xi)\, \xi\, y(\xi)\, d\xi + c_1 x \tag{3}$$

Now, using y(1) = 0 in (3), we get

$$0 = \frac{1}{2} - \int_0^1 (1-\xi)\ \xi\ y(\xi)\ d\xi + c_1$$

$$\Rightarrow \qquad c_1 = \int_0^1 \xi(1-\xi)\ \ y(\xi)\ d\xi - \frac{1}{2} \qquad (4)$$

Putting value of c_1 in (3), we get

$$y(x) = \frac{x^2}{2} - \int_0^x (x-\xi)\ \xi\ y(\xi)\ d\xi + x\int_0^1 \xi(1-\xi)\ \ y(\xi)\ d\xi - \frac{x}{2}$$

or $$y(x) = \frac{x^2}{2} - \frac{x}{2} - \int_0^x \xi(x-\xi)\ \ y(\xi)\ d\xi + x\int_0^1 \xi(1-\xi)\ \ y(\xi)\ d\xi$$

To express this in standard form, we split the second integral into two integrals, as follows

$$y(x) = \frac{x^2}{2} - \frac{x}{2} - \int_0^x \xi(x-\xi)\ \ y(\xi)\ d\xi + x\int_0^x \xi(1-\xi)\ \ y(\xi)\ d\xi\ +\ x\int_x^1 \xi(1-\xi)\ \ y(\xi)\ d\xi$$

or $$y(x) = \frac{x^2}{2} - \frac{x}{2} + \int_0^x (x - x\xi - x + \xi)\ \xi\ y(\xi)\ d\xi + x\int_x^1 (1-\xi)\ \xi\ y(\xi)\ d\xi$$

or $$y(x) = \frac{x^2}{2} - \frac{x}{2} + \int_0^x \xi(1-x)\xi\ \ y(\xi)\ d\xi + \int_x^1 \xi(1-\xi)x\ \ y(\xi)\ d\xi$$

or $$y(x) = f(x) + \int_0^1 \xi\, K(x,\xi)\ y(\xi)\ d\xi \quad \text{where } f(x) = \frac{x^2}{2} - \frac{x}{2}$$

and $$K(x, \xi) = \begin{cases} (1-x)\xi & if \quad 0 \le \xi \le x \\ (1-\xi)x & if \quad x \le \xi \le 1 \end{cases}.$$

Hence the solution.

2.2.2. Example. Reduce the boundary value problem,

$$y'' + A(x)\ y' + B(x)y = g(x),\ a \le x \le b,\ y(a) = c_1,\ y(b) = c_2$$

to a Fredholm integral equation.

Solution : Given differential equation is

$$y'' + A(x)\ y' + B(x)\ y = g(x)$$

$$\Rightarrow \qquad y'' = -A(x)\ y' - B(x)y + g(x) \qquad (1)$$

Integrating w.r.t. x from a to x, we get

$$\frac{dy}{dx} = -\int_a^x A(\xi)\ y'(\xi)\ d\xi - \int_a^x B(\xi)\ y(\xi)\ d\xi + \int_a^x g(\xi)\ d\xi + \alpha_1$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left[A(\xi)\, y(\xi)\ \right]_a^x + \int_a^x A'(\xi)\, y(\xi)\, d\xi - \int_a^x B(\xi)\, y(\xi)\, d\xi + \int_a^x g(\xi)\, d\xi + \alpha_1$$

$$\Rightarrow \quad \frac{dy}{dx} = \int_a^x [A'(\xi) - B(\xi)]\, y(\xi) d\ (\xi) + \int_a^x g(\xi)\, d\xi - A(x)\, y(x) + A(a)\ c_1 + \alpha_1$$

Again integrating over a to x,

$$y(x) = \int_a^x (x-\xi)[A'(\xi) - B(\xi)]\, y(\xi)\, d\xi + \int_a^x (x-\xi)\, g(\xi)\, d\xi - \int_a^x A(\xi)\, y(\xi)\, d\xi + (x-a)[\alpha_1 + A(a)\ \ c_1] + \alpha_2 \tag{2}$$

Applying first boundary condition, $y(a) = c_1$, we get $\alpha_2 = c_1$

Again applying second boundary condition, $y(b) = c_2$, we have

$$c_2 = \int_a^b (b-\xi)[A'(\xi) - B(\xi)]\, y(\xi)\, d\xi + \int_a^b (b-\xi)\, g(\xi)\, d\xi - \int_a^b A(\xi)\, y(\xi)\, d\xi + (b-a)[\alpha_1 + c_1 A(a)] + c_1$$

$$\Rightarrow \alpha_1 + c_1 A(a) = \frac{1}{b-a}\left[c_2 - c_1 - \int_a^b \left[(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)\right]\, y(\xi)\, d\xi - \int_a^b (b-\xi)\, g(\xi)\, d\xi \right]$$

$$\Rightarrow \begin{aligned} a_1 + c_1 A(a) = \frac{1}{b-a}\Bigg\{ & c_2 - c_1 - \int_a^x \left[(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)\right]\, y(\xi)\, d\xi \\ & - \int_x^b \left[(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)\right]\, y(\xi)\, d\xi - \int_a^b (b-\xi)\, g(\xi)\, d\xi \Bigg\} \end{aligned}$$

Putting this value of $\alpha_1 + c_1 A(a)$ in (2), we obtain

$$y(x) = c_1 + \int_a^x (x-\xi)\, g(\xi)\, d\xi + \frac{x-a}{b-a}\left[c_2 - c_1 - \int_a^b (b-\xi)\, g(\xi)\, d\xi \right]$$

$$+ \int_a^x \left[(x-\xi)\left\{A'(\xi) - B(\xi)\right\} - A(\xi)\right] y(\xi)\, d\xi - \int_a^x \frac{x-a}{b-a}\left[(b-\xi)\left\{A'(\xi) - B(\xi)\right\} - A(\xi)\right]\, y(\xi)\, d\xi$$

$$- \frac{x-a}{b-a}\int_x^b \left[(b-\xi)\left\{A'(\xi) - B(\xi)\right\} - A(\xi)\right] y(\xi)\, d\xi$$

or $y(x) = f(x) + \int_a^x \left[\left\{(x-\xi)-\frac{(x-a)(b-\xi)}{b-a}\right\}\{A'(\xi)-B(\xi)\}+A(\xi)\left\{-1+\frac{x-a}{b-a}\right\}\right]y(\xi)\,d\xi$

$$-\frac{x-a}{b-a}\int_x^b \left[(b-\xi)\{A'(\xi)-B(\xi)\}-A(\xi)\right]y(\xi)\,d\xi$$

Now, $(x-\xi)-\frac{(x-a)(b-\xi)}{b-a}=\frac{(x-b)(\xi-a)}{b-a}$ and $-1+\frac{x-a}{b-a}=\frac{x-b}{b-a}$

Thus, the above equation becomes

$$y(x)=f(x)+\frac{x-b}{b-a}\int_a^x \left[A(\xi)-(a-\xi)\{A'(\xi)-B(\xi)\}\right]y(\xi)\,d\xi -$$

$$\frac{x-a}{b-a}\int_a^b \left[A(\xi)-(b-\xi)\{A'(\xi)-B(\xi)\}\right]y(\xi)\,d\xi$$

or $\quad y(x) = f(x) + \int_a^b K(x,\ \xi)\,y(\xi)\,d\xi$

where $f(x) = c_1 + \int_a^x (x-\xi)\,g(\xi)\,d\xi+\frac{x-a}{b-a}\left[c_2-c_1-\int_a^b (b-\xi)\,g(\xi)\,d\xi\right]$

and $\quad K(x,\ \xi) = \begin{bmatrix} \frac{x-b}{b-a}\left[A(\xi)-(a-\xi)\{A'(\xi)-B(\xi)\}\right] & x > \xi \\ \frac{x-a}{b-a}\left[A(\xi)-(b-\xi)\{A'(\xi)-B(\xi)\}\right] & x < \xi \end{bmatrix}$

This completes the solution.

2.2.3. Example. Convert the Fredholm integral equation

$u(x) = \lambda\int_0^1 K(x,\ t)\,u(t)\,dt$ where $K(x, t) = \begin{cases} x(1-t) & 0 \le x \le t \\ t(1-x) & t \le x \le 1 \end{cases}$ into the boundary value problem $u'' + \lambda u = 0,\ u(0)= 0,\ u(1) = 0.$

Solution. Write $\quad u(x) = \lambda\left[\int_0^x K(x,\ t)\,u(t)\,dt+\int_x^1 K(x,\ t)\,u(t)\,dt\right]$

$$= \lambda\left[\int_0^x t(1-x)\,u(t)\,dt+\int_x^1 x(1-t)\,u(t)\,dt\right]$$

$$= \lambda\int_0^x t(1-x)\, u(t)\, dt + \lambda\int_x^1 x(1-t)\, u(t)\, dt \qquad (1)$$

Differentiating (1), w.r.t. x and using Leibnitz formula

$$\frac{du}{dx} = \lambda\int_0^x -t\, u(t)\, dt + \lambda x(1-x)\, u(x) + \lambda\int_x^1 (1-t)\, u(t)\, dt - \lambda x(1-x)\, u(x)$$

So, $$\frac{du}{dx} = \lambda\int_0^x -t\, u(t)\, dt + \lambda\int_x^1 (1-t)\, u(t)\, dt$$

Again differentiating w.r.t. x and using Leibnitz rule :

$$\frac{d^2u}{dx^2} = \lambda\int_0^x 0.(-t)\, u(t)\, dt + \lambda(-x)\, u(x) + \lambda\int_x^1 0.(1-t)\, u(t)\, dt - \lambda(1-x)\, u(x)$$

$$= -\lambda\, u(x)$$

$$\Rightarrow \quad \frac{d^2u}{dx^2} + \lambda\, u(x) = 0$$

Also, from (1), we have, $u(0) = 0 = u(1)$

Hence the solution.

2.2.4. Exercise. Reduce the following boundary value problems to Fredholm integral equation.

1. $y'' - \lambda y = 0$, $a < x < b$, $y(a) = 0 = y(b)$

Answer. $y(x) = \lambda\int_a^b K(x, \xi)\, y(\xi)\, d\xi$ where $K(x, \xi) = \begin{cases} \dfrac{(x-b)(\xi-a)}{b-a} & \text{if} \quad a \le \xi \le x \\ \dfrac{(x-a)(\xi-b)}{b-a} & \text{if} \quad x \le \xi \le b \end{cases}$.

2. $y'' + \lambda y = 0$, $y(0) = 0$, $y(l) = 0$

Answer. $y(x) = \lambda\int_0^l K(x, \xi)\, y(\xi)\, d\xi$ where $K(x, \xi) = \begin{cases} \dfrac{\xi(l-x)}{l} & \text{if} \quad 0 \le \xi \le x \\ \dfrac{x(l-\xi)}{l} & \text{if} \quad x \le \xi \le l \end{cases}$.

3. $y'' + \lambda y = x$; $y(0) = 0$, $y'(1) = 0$

Answer. $y(x) = \dfrac{1}{6}(x^3 - 3x) + \lambda\int_0^1 K(x, \xi)\, y(\xi)\, d\xi$ where $K(x, \xi) = \begin{cases} x & , \quad x > \xi \\ \xi & , \quad x < \xi \end{cases}$.

4. $y'' + \lambda y = 2x + 1$, $y(0) = y'(1)$, $y'(0) = y(1)$

Answer. $y(x) = f(x) + \lambda \int_0^1 K(x, \xi)\, y(\xi)\, d\xi$ where $f(x) = \frac{1}{6}\left[2x^3 + 3x^2 - 17x - 5\right]$ and

$$K(x, \xi) = \begin{cases} 1 + x(1-\xi) & \xi < x \\ (1-\xi) + (2-\xi)x & \xi > x \end{cases}.$$

5. $y'' + \lambda y = e^x$ $y(0) = y'(0)$, $y(1) = y'(1)$.

Answer. $y(x) = e^x + \lambda \int_0^1 K(x, \xi)\, y(\xi)\, d\xi$ where $K(x, \xi) = \begin{cases} -(1+\xi)x & ; \quad x > \xi \\ -(1+x)\xi & ; \quad x < \xi \end{cases}$.

2.3. Solution of Fredholm Integral Equation. Consider a Fredholm integral equation of second kind

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u(\xi)\, d\xi \qquad (1)$$

We define an integral operator,

$$k[\phi(x)] = \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi$$

$k^2[\phi(x)] = k\left[k\{\phi(x)\}\right]$ and so on.

Then, (1) can be written as

$$u(x) = f(x) + \lambda\, k[u(x)].$$

2.3.1. Theorem. If the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u(\xi)\, d\xi \qquad (1)$$

is such that

(i) $K(x, \xi)$ is a non – zero real valued continuous function in the rectangle $R = I \times I$, where $I = [a, b]$ and $|K(x, \xi)| < M$ in R .

(ii) f(x) is a non-zero real valued and continuous function on I.

(iii) λ is a constant satisfying the inequality, $|\lambda| < \frac{1}{M(b-a)}$.

Then (1) has one and only one continuous solution in the interval I and this solution is given by the absolutely and uniformly convergent series $u(x) = f(x) + \lambda\, k[f(x)] + \lambda^2 k^2 [f(x)] + \ldots$ to ∞.

Proof. We prove the result by the method of successive approximation. In this method we choose any continuous function say $u_0(x)$ defined on I as the zeroth approximation.

Then the first approximation, say $u_1(x)$, is given

$$u_1(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u_0(\xi)\, d\xi \tag{2}$$

By substituting this approximation into R.H.S. of (1), we obtain next approximation, $u_2(x)$. Continuing like this, we observe that the successive approximations are determined by the recurrence formula

$$u_n(x) = f(x) + \lambda \int_a^b K(x, \xi)\, u_{n-1}(\xi)\, d\xi \tag{3}$$

$$= f(x) + \lambda\, k\left[u_{n-1}(x)\right]$$

$$= f(x) + \lambda k\left[f(x) + \lambda k\left\{u_{n-2}(x)\right\}\right]$$

$$= f(x) + \lambda k\left[f(x)\right] + \lambda^2 k^2\left[f(x) + \lambda k\left\{u_{n-3}(x)\right\}\right]$$

Hence, $u_n(x) = f(x) + \lambda k\left[f(x)\right] + \lambda^2 k^2\left[f(x)\right] + \ldots + \lambda^{n-1} k^{n-1}\left[f(x)\right] + R_n(x)$,

where $R_n(x) = \lambda^n\, k^n[u_0(x)]$.

As $u_0(x)$ is continuous, it is bounded that is, $|u_0(x)| \le U$ in I

Now, $|R_n(x)| = |\lambda|^n \left[\int_a^b K(x, t) \int_a^b K(t, t_1) \ldots \int_a^b K(t_{n-2}, t_{n-1}) u_0(t_{n-1}) dt_{n-1} \ldots dt\right]$

$$\le |\lambda|^n\; M^n\; U\; (b-a)^n$$

$$= U\left[|\lambda|\; M\,(b-a)\right]^n \to 0 \text{ as } n \to \infty \quad \left(\text{Since}, |\lambda| < \frac{1}{M(b-a)}\right)$$

$$\Rightarrow \quad \lim_{n \to \infty} R_n(x) = 0$$

Thus, $\lim_{n \to \infty} u_n(x) = u(x) = f(x) + \lambda\, k\, f(x) + \lambda^2 k^2\, f(x) + \ldots$ to ∞

This can be easily verified by the virtue of M – test that the above series is absolutely and uniformly convergent in I.

Uniqueness. Let $v(x)$ be another solution of given integral equation then by choosing $u_0(x) = v(x)$, we get

$$u_n(x) = v(x) \text{ for all } n$$

$$\Rightarrow \quad \lim_{n \to \infty} u_n(x) = v(x) \Rightarrow u(x) = v(x).$$

This completes the proof.

2.3.2. Example. Find the first two approximation of the solution of Fredholm integral equation.

$$u(x) = 1 + \int_0^1 K(x, \xi)\, u(\xi)\, d\xi \text{ where } K(x, \xi) = \begin{cases} x & 0 \le x \le \xi \\ \xi & \xi \le x \le 1 \end{cases}.$$

Solution. Let $u_0(x) = 1$ be the zeroth approximation. Then first approximation is given by

$$u_1(x) = 1 + \int_0^1 K(x\,,\,\xi)\, u_0(\xi)\, d\xi$$

$$= 1 + \int_0^x K(x\,,\,\xi)\; d\xi + \int_x^1 K(x\,,\,\xi)\; d\xi \qquad = 1 + \int_0^x \xi\, d\xi + \int_x^1 x\, d\xi$$

$$= 1 + \frac{x^2}{2} + x(1-x) \;= 1 + x - \frac{x^2}{2}$$

Now,
$$u_2(x) = 1 + \int_0^1 K(x\,,\,\xi)\; u_1(\xi)\, d\xi$$

$$= 1 + \int_0^1 K(x\,,\,\xi)\left(1+\xi-\frac{\xi^2}{2}\right) d\xi$$

$$= 1 + \int_0^x \xi\left(1+\xi-\frac{\xi^2}{2}\right) d\xi + x\int_x^1\left(1+\xi-\frac{\xi^2}{2}\right) d\xi$$

$$= 1 + \frac{4}{3}x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\;.$$

2.4. Resolvent kernel for Fredholm integral equation. Consider the Fredholm integral equation

$$u(x) = f(x) + \lambda\int_a^b K(x\,,\,\xi)\, u(\xi)\, d\xi \qquad (1)$$

The iterated kernels are defined by $K_1(x,\ \xi) = K(x,\ \xi)$, and

$$K_{n+1}(x,\xi) = \int_a^b K(x\,,\,t)\; K_n(t\,,\,\xi)\, dt\,,\ n = 1, 2, 3, \ldots$$

and the solution of (1) is given by :

$$u(x) = f(x) + \lambda\int_a^b R(x\,,\,\xi : \lambda)\; f(\xi)\, d\xi$$

where
$$R(x,\ \xi\ :\ \lambda) = K_1 + \lambda K_2 + \lambda^2 K_3 + \ldots \text{ to } \infty \qquad = \sum_{n=1}^{\infty} \lambda^{n-1}\; K_n(x,\ \xi)$$

2.4.1. Neumann series. The infinite series $K_1 + \lambda K_2 + \lambda^2 K_3 + \ldots\ldots$ is called Neumann series.

2.4.2. Resolvent Kernel. The function $R(x,\ \xi\ :\ \lambda)$ is called Resolvent Kernel.

2.4.3. Example. Obtain the Resolvent kernel associated with the kernel $K(x,\ \xi) = 1-3x\xi$ in the interval (0, 1) and solve the integral equation $u(x) = 1+\lambda\int_0^1 (1-3x\xi)\; u(\xi)\, d\xi$.

Solution. Here, $K(x, \xi) = 1-3x\xi$. We know that the iterated kernels are given by the relation,

$$K_1(x, \xi) = K(x, \xi)$$

and $$K_{n+1}(x, \xi) = \int_a^b K(x, t)\ K_n(t, \xi)\, dt$$

Therefore, $$K_1(x, \xi) = 1-3x\xi$$

and $$K_2(x, \xi) = \int_0^1 K(x, t)\ K_1(t, \xi)\, dt$$

$$= \int_0^1 (1-3xt)\,(1-3t\xi)\, dt$$

$$= \int_0^1 (1-3t\xi-3xt+9xt^2\xi)\, dt$$

$$= \left[t-\frac{3t^2\xi}{2}-\frac{3xt^2}{2}+3xt^3\xi\right]_0^1$$

$$= 1-\frac{3}{2}\xi-\frac{3}{2}x+3x\xi$$

$$K_3(x, \xi) = \int_0^1 K(x, t)\ K_2(t, \xi)\, dt$$

$$= \int_0^1 (1-3xt)\left(1-\frac{3}{2}t-\frac{3}{2}\xi+3t\xi\right) dt$$

$$= \frac{1}{4}(1-3x\xi) \text{ (on solving)}$$

$$K_4(x, \xi) = \int_0^1 K(x, t)\ K_3(t, \xi)\, dt$$

$$= \frac{1}{4}\int_0^1 (1-3xt)\,(1-3t\xi)\, dt$$

$$= \frac{1}{4}\left[1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right]$$

The Resolvent Kernel $R(x, \xi : \lambda)$ is given by

$$R(x, \xi : \lambda) = K_1 + \lambda K_2 + \lambda^2 K_3 + \lambda^4 K_4 + \ldots$$

$$= (1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)+\frac{\lambda^2}{4}(1-3x\xi)+\frac{\lambda^3}{4}\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)+\ldots$$

$$= (1-3x\xi)\left(1+\frac{\lambda^2}{4}\right)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\left(1+\frac{\lambda^2}{4}\right)+\ldots$$

$$= \left(1+\frac{\lambda^2}{4}+\ldots\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

$$= \left(\frac{1}{1-\frac{\lambda^2}{4}}\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

$$= \left(\frac{4}{4-\lambda^2}\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

which provides the required result.

We know that the solution of an integral equation

$$u(x) = f(x) + \lambda\int_a^b K(x, \xi)\ u(\xi)\, d\xi \text{ is given by}$$

$$u(x) = f(x) + \lambda\int_a^b R(x, \xi : \lambda)\, f(\xi)\, d\xi$$

Here, $K(x, \xi) = (1-3x\xi)$. Then,

$$R(x, \xi : \lambda) = \left(\frac{4}{4-\lambda^2}\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

Thus, the solution of given integral equation is

$$u(x) = 1+\frac{4\lambda}{4-\lambda^2}\int_0^1\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right].1d\xi$$

$$= 1+\frac{4\lambda}{4-\lambda^2}\left[\xi-3x\frac{\xi^2}{2}+\lambda\left(\xi-\frac{3\xi^2}{4}-\frac{3x\xi}{2}+\frac{3x\xi^2}{2}\right)\right]_0^1$$

$$= 1+\frac{4\lambda}{4-\lambda^2}\left[1-\frac{3x}{2}+\lambda\left(1-\frac{3}{4}-\frac{3x}{2}+\frac{3x}{2}\right)\right]$$

$$= 1+\frac{4\lambda}{4-\lambda^2}\left(1-\frac{3x}{2}+\frac{\lambda}{4}\right) = \frac{4+4\lambda-6x\lambda}{4-\lambda^2},\ \lambda \neq \pm 2$$

This is the required solution of given integral equation.

2.4.4. Exercise. Determine the Resolvent Kernel associated with $K(x, \xi) = x\xi$ in the interval (0, 1) in the form of a power series in λ.

Answer. $R(x, \xi : \lambda) = \dfrac{3}{3-\lambda}x\xi$, $|\lambda| < 3$

2.4.5. Exercise. Solve the following integral equations by finding the resolvent kernel:

1. $u(x) = f(x) + \lambda\int_0^1 e^{(x-\xi)}u(\xi)\, d\xi$

Answer. $u(x) = f(x) + \dfrac{\lambda}{1-\lambda}\int_0^1 e^{(x-\xi)}f(\xi)\, d\xi$.

2. $u(x) = 1 + \lambda \int_0^1 xe^{\xi}\, u(\xi)\, d\xi$

Answer. $u(x) = 1 + \dfrac{\lambda x}{1-\lambda}(e-1)$

3. $u(x) = x + \lambda\int_0^1 xe^{\xi}\, u(\xi)\, d\xi$

Answer. $u(x) = x + \dfrac{\lambda x}{1-x}$

4. $u(x) = x + \lambda\int_0^1 x\xi\, u(\xi)\, d\xi$

Answer. $u(x) = x + \dfrac{\lambda x}{3-\lambda}$.

2.5. Separable kernel. A kernel $K(x, \xi)$ of an integral equation is called separable if it can be expressed in the form

$$K(x, \xi) = \sum_{i=1}^{n} a_i(x)\, b_i(\xi) = a_1(x)\, b_1(\xi) + a_2(x)\, b_2(\xi) + \ldots + a_n(x)\, b_n(\xi)$$

For example, (a) $e^{x-\xi} = e^x . e^{-\xi} = a_1(x)\, b_1(\xi)$, n = 1

(b) $x-\xi = x.1 + 1(-\xi) = a_1(x)\, b_1(\xi) + a_2(x)\, b_2(\xi)$, n = 2

(c) Similarly, $\sin(x + \xi)$, $1 - 3x\xi$ are separable kernels.

(d) x^{ξ}, $\sin(x\xi)$ are non – separable kernels.

2.5.1. Method to solve Fredholm integral equation of second kind with separable kernel.

Let the given integral equation be

$$u(x) = f(x) + \lambda\int_a^b K(x, \xi)\, u(\xi)\, d\xi \tag{1}$$

where

$$K(x, \xi) = \sum_{i=1}^{n} a_i(x)\, b_i(\xi) \tag{2}$$

Thus, (1) can be written as

$$u(x) = f(x) + \lambda \int_a^b \left[\sum_{i=1}^{n} a_i(x)\, b_i(\xi) \right] u(\xi)\, d(\xi)$$

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} a_i(x) \left[\int_a^b b_i(\xi)\, u(\xi)\, d(\xi) \right]$$

$$= f(x) + \lambda \left[c_1 a_1(x) + c_2 a_2(x) + \ldots + c_n a_n(x) \right] \quad (3)$$

where $$C_k = \int_a^b b_k(\xi)\, u(\xi)\, d\xi = \int_a^b b_k(x)\, u(x)\, dx \quad (4)$$

Here, (3) gives the solution of given Fredholm integral (1) provided the constants $c_1, c_2, \ldots, c_n$ are determined.

For this, we multiply (3) both sides by $b_i(x)$ and then integrating w.r.t. x from a to b, we find

$$\int_a^b b_i(x)\, u(x)\, dx = \int_a^b f(x)\, b_i(x)\, dx + \lambda \sum_{k=1}^{n} C_k \int_a^b b_i(x)\, a_k(x)\, dx \quad \text{for } i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad c_i = f_i + \lambda \sum_{k=1}^{n} \alpha_{ik} C_k \quad (5)$$

where $$f_i = \int_a^b f(x)\, b_i(x)\, dx \quad \text{and} \quad \alpha_{ik} = \int_a^b b_i(x)\, a_k(x)\, dx \quad (6)$$

Now, from (5)

$$c_1 = f_1 + \lambda \left[\alpha_{11} c_1 + \alpha_{12} c_2 + \ldots + \alpha_{1n} c_n \right]$$

$$c_2 = f_2 + \lambda \left[\alpha_{21} c_1 + \alpha_{22} c_2 + \ldots + \alpha_{2n} c_n \right]$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$c_n = f_n + \lambda \left[\alpha_{n1} c_1 + \alpha_{n2} c_2 + \ldots + \alpha_{nn} c_n \right]$$

In matrix form, $C = F + \lambda AC$ or $(I - \lambda A)\, C = F$ (7)

where $$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},\ F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix},\ A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} \end{bmatrix}$$

Let $$|I - \lambda A| = \Delta(\lambda) \quad (8)$$

Now, we discuss the various cases

Case I. When $f(x) \neq 0$ and $F \neq 0$, that is, both integral equation as well as matrix equation are non – homogeneous. Then, (7) has a unique solution if and only if $\Delta(\lambda) \neq 0$

If $\Delta(\lambda) = 0$ for some value of λ, then (7) has no solution or infinite solutions.

Case II. When f(x) = 0 that is, the Fredholm integral equation is homogeneous. In this case f_i = o for all i and consequently F = 0. Thus, (7) reduces to :

$$(I - \lambda A)C = 0 \qquad (9)$$

Subcase (a). If $\Delta(\lambda) \neq 0$, then (9) has the trivial solution, C = 0 that is, $C_i = 0$ for all i .

Hence the (3) becomes, u(x) = 0 which is the solution of given integral equation.

Subcase (b). If $\Delta(\lambda_0) = 0$ for some scalar λ_0, then (9) has infinitely many solutions. Consequently, the Fredholm integral equation $u(x) = \lambda_0 \int_a^b K(x, \xi)\, u(\xi)\, d\xi$ has infinitely many solutions.

Case III. When $f(x) \neq 0$ but F = 0. In this case also,

$$(I - \lambda A)\, C = 0 \qquad (10)$$

Subcase (a). If $\Delta(\lambda) \neq 0$, then (10) has only trivial solution C = 0 that is, $C_i = 0$ for all i.

Hence the required solution of given equation becomes

$$u(x) = f(x) + 0 = f(x)$$

Sub case (b). If $\Delta(\lambda_0) = 0$ for some scalar $\lambda = \lambda_0$, then (10) has infinitely many solutions, therefore the given equation, $u(x) = f(x) + \lambda_0 \int_a^b K(x, \xi)\, u(\xi)\, d\xi$ has infinitely many solutions.

2.5.2. Eigen values and Eigen functions. The values of λ for which $\Delta(\lambda) = 0$ are called eigen values (or characteristic numbers) of Fredholm integral equation. The non – trivial solution corresponding to eigen values are called eigen functions (or characteristic functions).

Remark. Separable kernels are also known as degenerate kernels.

2.5.3. Example. Solve the integral equation and discuss all its possible cases with the method of separable kernels

$$u(x) = f(x) + \lambda \int_0^1 (1 - 3x\xi)\, u(\xi)\, d\xi .$$

Solution. The given equation is

$$u(x) = f(x) + \lambda \int_0^1 (1 - 3x\xi)\, u(\xi)\, d\xi \qquad (1)$$

$$\Rightarrow \quad u(x) = f(x) + \lambda\,[C_1 - 3xC_2] \qquad (2)$$

where, $$c_1 = \int_0^1 u(\xi)\,d\xi \qquad (3)$$

and $$c_2 = \int_0^1 \xi\,u(\xi)\,d\xi \qquad (4)$$

c_1 and c_2 are constants to be determined.
Integrating (2), w.r.t.x over the limit 0 to 1.

$$\int_0^1 u(x)\,dx = \int_0^1 f(x)\,dx + \lambda\int_0^1 (c_1 - 3xc_2)\,dx$$

$$\Rightarrow \quad c_1 = \int_0^1 f(x)\,dx + \lambda\left(c_1 - \frac{3}{2}c_2\right) \qquad \text{[Using (3)]}$$

or $$(1-\lambda)\,c_1 + \frac{3}{2}\lambda\,c_2 = f_1 \qquad (5)$$

where $$f_1 = \int_0^1 f(x)\,dx$$

Now multiplying (2) with x and integrating w.r.t. x between limits 0 and 1, we get

$$\int_0^1 x\,u(x)\,dx = \int_0^1 x\,f(x)\,dx + \lambda\int_0^1 \left(c_1 x - 3x^2 c_2\right)\,dx$$

or $$c_2 = f_2 + \lambda\left[c_1\frac{x^2}{2} - x^3 c_2\right]_0^1 \qquad \text{[Using (4)]}$$

$$= f_2 + \lambda\left(\frac{c_1}{2} - c_2\right)$$

or $$-\frac{\lambda}{2}c_1 + (1+\lambda)\,c_2 = f_2 \qquad (6)$$

where $f_2 = \int_0^1 x\,f(x)\,dx$

From (5) and (6), we get, $\Delta(\lambda) = \begin{vmatrix} 1-\lambda & \frac{3\lambda}{2} \\ -\frac{\lambda}{2} & 1+\lambda \end{vmatrix} = 1-\lambda^2 + \frac{3\lambda^2}{4} = 1 - \frac{\lambda^2}{4}$

or $$\Delta(\lambda) = \frac{4-\lambda^2}{4}$$

Now, (5) and (6) can be written as

$$(I - \lambda A)\, C = f$$

where $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, $F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

Also, $|I - \lambda A| = \Delta(\lambda)$.

Case I. When $f(x) \neq 0$ and $F \neq 0$ then equations (5) and (6) has a unique solution if $\Delta(\lambda) \neq 0$, that is, $\lambda \neq 2, -2$. When $\lambda = 2$ or -2, then these equations have either no solution or infinite many solutions.

(i) $\lambda = 2$

Then, (5) and (6) reduce to

$$\left.\begin{aligned} -c_1 + 3c_2 &= f_1 \\ -c_1 + 3c_2 &= f_2 \end{aligned}\right] \quad (7)$$

These equation have no solution if $f_1 \neq f_2$ and have infinitely many solutions when $f_1 = f_2$, that is,

$$\int_0^1 f_1(x)\,dx = \int_0^1 x\, f(x)\,dx$$

or $$\int_0^1 (1 - x)\, f(x)\,dx = 0$$

Thus, the solution of given integral equation is

$$\begin{aligned} u(x) &= f(x) + 2[c_1 a_1(x) + c_2 a_2(x)] \\ &= f(x) + 2[c_1.1 + c_2(-3x)] = f(x) + 2[3c_2 - f_1 - 3xc_2] \qquad \text{[From (7)]} \\ &= f(x) + 6c_2(1 - x) - 2f_1 \end{aligned}$$

or $$u(x) = f(x) + 6c_2(1 - x) - 2\int_0^1 f(x)\,dx \quad \text{where } c_2 \text{ is arbitrary.}$$

(ii) $\lambda = -2$

As done above, the solution is given by

$$u(x) = f(x) - 2(1 - 3x)c_2 - 2\int_0^1 x\, f(x)\,dx$$

Case II. When $f(x) = 0$, $F = 0$

In this case, the equations (5) and (6) becomes :

$$(1-\lambda)\, c_1 + \frac{3\lambda}{2}\, c_2 = 0$$
$$\frac{-\lambda}{2}\, c_1 + (1+\lambda)\, c_2 = 0 \qquad (8)$$

If $\lambda \neq 2, -2$, then system has only trivial solution $c_1 = 0 = c_2$. Thus $u(x) = 0$ is the solution of given integral equation.

(i) $\lambda = 2$

Then, (8) becomes

$$-c_1 + 3c_2 = 0 \quad \Rightarrow \quad c_1 = 3c_2$$

Thus the solution of given integral equation is

$$u(x) = 0 + 2(3c_2 - 3xc_2) = 6c_2(1-x).$$

(ii) $\lambda = -2$

Then, (8) becomes

$$c_1 - c_2 = 0 \Rightarrow c_1 = c_2$$

Thus the solution is

$$u(x) = 0 - 2[c_2 - 3xc_2] = 2c_2(3x-1).$$

Case III. When $f(x) \neq 0$ and $F = 0$

If $\lambda \neq 2, -2$, the system (8) has only trivial solution $c_1 = c_2 = 0$ and therefore $u(x) = f(x)$ is the solution.

(i) $\lambda = 2$

Then $c_1 = 3c_2$ and the solution is

$$u(x) = f(x) + 2(3c_2 - 3xc_2) = f(x) + 6c_2(1-x).$$

(ii) $\lambda = -2$

Then $c_1 = c_2$ and the solution is

$$u(x) = f(x) - 2[c_2 - 3xc_2] = f(x) - 2c_2(1-3x)$$

This completes the solution.

2.5.4. Example : Find the eigen values and eigen functions of the integral equation

$$u(x) = \lambda \int_0^{2\pi} \sin(x+t)\, u(t)\, dt\ .$$

Answer. Eigen values are $\lambda = \pm\frac{1}{\pi}$. For $\lambda = \frac{1}{\pi}$, eigen function is u(x) = A(sinx + cosx), where A = $\frac{c_1}{\pi}$ and for $\lambda = -\frac{1}{\pi}$, eigen function is u(x) = B(sinx − cosx), where B = $\frac{c_2}{\pi}$.

2.5.5. Exercise. Solve the integral equations by the method of degenerate kernel:

1. $u(x) = x + \lambda\int_0^1 (xt^2 + x^2t)\, u(t)\, d(t)$

Answer. $u(x) = \frac{(240-60\lambda)\, x + 80\lambda x^2}{240-120\lambda-\lambda^2}$.

2. $u(x) = e^x + \lambda\int_0^1 2e^x e^t\, u(t)\, dt$

Answer. $u(x) = \frac{e^x}{1-\lambda(e^2-1)}$.

2.6. Symmetric kernel. The kernel K(x, ξ) of an integral equation is said to be symmetric if

$$K(x, \xi) = K(\xi, x) \text{ for all } x \text{ and } \xi.$$

2.6.1. Orthogonality. Two functions $\phi_1(x)$ and $\phi_2(x)$ continuous on an interval (a, b) are said to be orthogonal if $\int_a^b \phi_1(x)\, \phi_2(x)\, dx = 0$.

2.6.2. Theorem. For the Fredholm integral equation $y(x) = \lambda\int_a^b K(x, \xi)\, y(\xi)\, d\xi$ with symmetric kernel, prove that :

(i) The eigen functions corresponding to two different eigen values are orthogonal over (a, b).

(ii) The eigen values are real.

Proof. (i) Let λ_1 and λ_2 be two different eigen values of given integral equation

$$y(x) = \lambda\int_a^b K(x, \xi)\, y(\xi)\, d\xi \tag{1}$$

w.r.t. eigen functions $y_1(x)$ and $y_2(x)$. We have to show that

$$\int_a^b y_1(x)\, y_2(x)\, dx = 0 \tag{2}$$

By definition we have,

$$y_1(x) = \lambda_1 \int_a^b K(x,\xi)\, y_1(\xi)\, d\xi \tag{3}$$

$$y_2(x) = \lambda_2 \int_a^b K(x,\xi)\, y_2(\xi)\, d\xi \tag{4}$$

Multiplying (3) by $y_2(x)$ and then integrating w.r.t. x over the interval a to b, we find

$$\int_a^b y_1(x)\, y_2(x)\, dx = \lambda_1 \int_a^b y_2(x) \left[\int_a^b K(x,\xi)\, y_1(\xi) d\xi \right] dx$$

Interchanging the order of integration

$$\int_a^b y_1(x)\, y_2(x)\, dx = \lambda_1 \int_a^b y_1(\xi) \left[\int_a^b K(x,\xi)\, y_2(x) dx \right] d\xi$$

$$= \lambda_1 \int_a^b y_1(\xi) \left[\int_a^b K(\xi, x)\, y_2(x)\, dx \right] d\xi \quad [\text{Since } K(x, \xi) = K(\xi, x)]$$

$$= \lambda_1 \int_a^b y_1(\xi) \frac{y_2(\xi)}{\lambda_2} d\xi \qquad [\text{By (4)}]$$

$$= \frac{\lambda_1}{\lambda_2} \int_a^b y_1(x)\, y_2(x)\, dx$$

$$\Rightarrow \quad \left(1 - \frac{\lambda_1}{\lambda_2}\right) \int_a^b y_1(x)\, y_2(x)\, dx = 0$$

$$\Rightarrow \quad \int_a^b y_1(x)\, y_2(x)\, dx = 0, \qquad (\lambda_1 \neq \lambda_2)$$

(ii) If possible, we assume on the contrary that there is an eigen value λ_0 (say) which is not real.

So, $\quad \lambda_0 = \alpha_0 + i\beta_0,\ \beta_0 \neq 0$ (5)

where α_0 and β_0 are real.

Let $y_0(x) \neq 0$ be the corresponding eigen function. Then

$$y_0(x) = \lambda_0 \int_a^b K(x,\xi)\, y_0(\xi)\, d\xi \tag{6}$$

We claim that the eigen function $y_0(x)$ corresponding to a non real eigen value λ_0 is not real valued. If $y_0(x)$ is real valued, then separating the real and imaginary parts in (6), we get

$$y_0(x) = \alpha_0 \int_a^b K(x, \xi)\ y_0(\xi)\ d\xi \qquad (7)$$

and
$$0 = \beta_0 \int_a^b K(x, \xi)\ y_0(\xi)\ d\xi \qquad (8)$$

$$\Rightarrow \quad \int_a^b K(x, \xi)\ y_0(\xi)\ d\xi = 0, \ (\beta_0 \neq 0)$$

Hence from (7), we get $y_0(x) = 0$, a contradiction. Thus $y_0(x)$ cannot be a real valued function.

Let us consider

$$y_0(x) = \alpha(x) + i\beta_0(x), \ \beta(x) \neq 0 \qquad (9)$$

Changing i to $-i$ in (6), we obtain

$$\overline{y_0(x)} = \overline{\lambda_0} \int_a^b K(x, \xi)\ \overline{y_0(\xi)}\ d\xi \qquad (10)$$

This shows that $\overline{\lambda_0}$ is an eigen value with corresponding eigen function $\overline{y_0(x)}$. Since λ_0 is non – real by assumption. So λ_0 and $\overline{\lambda_0}$ are two different eigen values. Thus by part (i), we have

$$\int_a^b y_0(x)\ \overline{y_0(x)}\ dx = 0$$

$$\Rightarrow \quad \int_a^b |y_0(x)|^2\ dx = 0$$

$$\Rightarrow \quad \int_a^b |\alpha(x) + i\beta(x)|^2\ dx = 0$$

$$\Rightarrow \quad \int_a^b \left([\alpha(x)]^2 + [\beta(x)]^2 \right) dx = 0$$

$$\Rightarrow \quad \alpha(x) = \beta(x) = 0$$

$$\Rightarrow \quad y_0(x) = 0,$$

a contradiction because eigen functions are non – zero. This contradiction shows that our assumption that λ_0 is not real is wrong. Hence λ_0 must be real.

This completes the proof.

2.6.3. Fredholm Resolvent kernel expressed as a ratio of two series in λ.

Consider the Fredholm integral equation

$$u(x) = f(x) + \lambda\int_a^b K(x\,,\,\xi)\;u(\xi)\;d\xi \qquad (1)$$

The resolvent kernel of (1) is also given by

$$R\,(x,\ \xi : \lambda) = \frac{D(x\,,\,\xi : \lambda)}{D(\lambda)} \qquad [D(\lambda) \neq 0]$$

where, $\quad D(x,\ \xi : \lambda) = K(x,\ \xi) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\,\lambda^n\ B_n\,(x,\ \xi)$

and $\quad D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\,\lambda^n\ c_n$

where

$$B_n(x,\ \xi) = \underbrace{\int_a^b\int_a^b\ldots\ldots\int_a^b}_{n\ \text{times}} \begin{vmatrix} K(x\,,\,\xi) & K(x\,,\,t_1) & \cdots & K(x\,,\,t_n) \\ K(t_1\,,\,\xi) & K(t_1\,,\,t_1) & \cdots & K(t_1\,,\,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n\,,\,\xi) & K(t_n\,,\,t_1) & \cdots & K(t_n\,,\,t_n) \end{vmatrix} dt_1\ dt_2\ldots\ldots dt_n$$

and

$$c_n = \underbrace{\int_a^b\int_a^b\ldots\ldots\int_a^b}_{n\ \text{times}} \begin{vmatrix} K(t_1\,,\,t_1) & K(t_1\,,\,t_2) & \cdots & K(t_1\,,\,t_n) \\ K(t_2\,,\,t_1) & K(t_2\,,\,t_2) & \cdots & K(t_2\,,\,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n\,,\,t_1) & K(t_n\,,\,t_2) & \cdots & K(t_n\,,\,t_n) \end{vmatrix} dt_1\ dt_2\ldots\ldots dt_n$$

Note that determinant in c_n is obtained by just removing first row and first column from the determinant in B_n.

2.6.4. Fredholm Determinant. $D(x,\ \xi : \lambda)$ is called Fredholm minor and $D(\lambda)$ is called Fredholm determinant.

Remark.

1. After finding the resolvent kernel $R(x,\ \xi : \lambda)$ the solution of given integral equation is given by

$$u(x) = f(x) + \lambda\int_a^b R(x\,,\,\xi : \lambda)\ f(\xi)\ d\xi$$

2. This method cannot be used when $\lambda = 1$.

2.6.5. Example. Using the Fredholm determinant, find the resolvent kernel of

$$K(x,\ \xi) = 2x - \xi, \qquad 0 \leq x \leq 1,\ 0 \leq \xi \leq 1.$$

Solution. Here the kernel is

$$K(x, \xi) = 2x - \xi \tag{1}$$

The resolvent kernel $R(x, \xi : \lambda)$ is given by

$$R(x, \xi : \lambda) = \frac{D(x, \xi : \lambda)}{D(\lambda)}, \; D(\lambda) \neq 0 \tag{2}$$

where $D(x, \xi : \lambda) = K(x, \xi) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n B_n(x, \xi)$

and $D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n c_n \quad (3)$

where $B_n = \int_a^b \int_a^b \ldots \int_a^b \begin{vmatrix} K(x, \xi) & K(x, t_1) & \cdots & K(x, t_n) \\ K(t\;, \xi) & K(t_1, t_1) & \cdots & K(t_1, t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n, \xi) & K(t_n, t_1) & \cdots & K(t_n, t_n) \end{vmatrix} dt_1\, dt_2 \ldots\ldots dt_n$

and $\int_a^b \int_a^b \ldots \int_a^b \begin{vmatrix} K(t_1, t_1) & K(t_1, t_2) & \cdots & K(t_1, t_n) \\ K(t_2, t_1) & K(t_2, t_2) & \cdots & K(t_2, t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n, t_1) & K(t_n, t_2) & \cdots & K(t_n, t_n) \end{vmatrix} dt_1\, dt_2 \ldots\ldots dt_n$

Therefore, $B_1(x, \xi) = \int_0^1 \begin{vmatrix} 2x - \xi & 2x - t_1 \\ 2t_1 - \xi & 2t_1 - t_1 \end{vmatrix} dt_1$

$$= \int_0^1 \left(2xt_1 - \xi t_1 - 4xt_1 + 2t_1^2 + 2x\xi - \xi t_1\right) dt_1$$

$$= \int_0^1 \left(-2xt_1 - 2\xi t_1 + 2t_1^2 + 2x\xi\right) dt_1$$

$$B_1(x, \xi) = -x - \xi + \frac{2}{3} + 2x\xi$$

$$B_2(x, \xi) = \int_0^1 \int_0^1 \begin{vmatrix} 2x - \xi & 2x - t_1 & 2x - t_2 \\ 2t_1 - \xi & 2t_1 - t_1 & 2t_1 - t_2 \\ 2t_2 - \xi & 2t_2 - t_1 & 2t_2 - t_2 \end{vmatrix} dt_1 dt_2$$

which on solving gives, $B_2(x, \xi) = 0$

In general $B_n(x, \xi) = 0$ for all $n \geq 2$

Now, $$c_1 = \int_0^1 (2t_1 - t_1)\, dt_1 = \frac{1}{2}$$

$$c_2 = \int_0^1\int_0^1 \begin{vmatrix} 2t_1 - t_1 & 2t_1 - t_2 \\ 2t_2 - t_1 & 2t_2 - t_2 \end{vmatrix} dt_1 dt_2 = \frac{1}{3}$$

Now, since $B_n = 0$ for all $n \geq 2$

$\Rightarrow \quad c_n = 0$ for all $n \geq 3$

Thus, from (3), we get

$$D(x, \xi : \lambda) = (2x - \xi) + (-1)\ \lambda \left(2\xi x - x - \xi + \frac{2}{3}\right)$$

$$= 2x - \xi + \lambda \left(x + \xi - 2x\xi - \frac{2}{3}\right)$$

$$D(\lambda) = 1 + (-1)^1\ \lambda\, c_1 + \frac{(-1)^2}{2!}\ \lambda\, c_2 = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}$$

Hence the resolvent kernel is given by :

$$R(x, \xi : \lambda) = \frac{(2x-\xi) + \lambda\left((x + \xi - 2x\xi - \frac{2}{3}\right)}{1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}}$$

Hence the solution.

2.6.6. Exercise. Using Fredholm determinant, find the resolvent kernel of $K(x, \xi) = 1 + 3x\xi$.

Answer. $$R(x, \xi : \lambda) = \frac{(1 + 3x\xi) - \lambda\left(1 + 3\xi x - \frac{3\xi}{2} - \frac{3x}{2}\right)}{1 - 2\lambda + \frac{\lambda^2}{4}}.$$

2.7. Check Your Progress.

1. Solve the following integral equations by finding the resolvent kernel:

$$u(x) = f(x) + \lambda \int_0^1 e^{a(x^2 - \xi^2)}\, u(\xi)\, d\xi$$

Answer. $$u(x) = f(x) + \frac{\lambda}{1-\lambda} \int_0^1 e^{a(x^2 - \xi^2)}\, f(\xi)\, d\xi.$$

2. Solve the integral equations by the method of degenerate kernel:

$$u(x) = x + \lambda \int_0^1 (1 + x + t)\, u(t)\, dt$$

Answer. $u(x) = x + \dfrac{\lambda}{12 - 24\lambda - \lambda^2}[10 + (6 + \lambda)x]$.

2.8. Summary. In this chapter, various methods like successive approximations, successive substitutions, resolvent kernel are discussed to solve a Fredholm integral equation. Also it is observed that a Fredholm integral equation always transforms into a boundary value problem.

Books Suggested:

1. Jerri, A.J., Introduction to Integral Equations with Applications, A Wiley-Interscience Publication, 1999.
2. Kanwal, R.P., Linear Integral Equations, Theory and Techniques, Academic Press, New York.
3. Lovitt, W.V., Linear Integral Equations, McGraw Hill, New York.
4. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.
5. Gelfand, J.M., Fomin, S.V., Calculus of Variations, Prentice Hall, New Jersey, 1963.

3

Green Function

Structure

3.1. Introduction. This chapter contains methods to obtain Green function for a given non-homogeneous linear second order boundary value problem and reduction of boundary value problem to Fredholm integral equation with Green function as kernel.

31.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Construction of Green function.

(ii) Reduction of boundary value problem to Fredholm integral equation with Green function as kernel.

3.1.2. Keywords. Green function, Integral Equations, Boundary Conditions.

3.2. Construction of Green function. Consider a differential equation of order n

$$L(u) = p_0(x)\, u^n + p_1(x)\, u^{n-1} + p_2(x)\, u^{n-2} + \ldots\ldots + p_n(x)\, u = 0 \qquad (1)$$

where the functions $p_0(x), p_1(x)$, $p_2(x),\ldots\ldots, p_n(x)$ are continuous on [a, b], $p_0(x) \neq 0$ on [a, b], and the boundary conditions

$$V_k(u) = \alpha_k u(a) + \alpha_k^1 u'(a) + \alpha_k^2 u''(a) + \ldots\ldots\ldots + \alpha_k^{n-1} u^{n-1}(a)$$

$$+\beta_k u(b) + \beta_k^1 u'(b) + \beta_k^2 u''(b) + \ldots\ldots\ldots + \beta_k^{n-1} u^{n-1}(b) \qquad (2)$$

for k = 1, 2,…,n, where the linear forms $V_1, V_2,\ldots, V_n$ in $u(a), u'(a),\ldots, u^{n-1}(a), u(b), u'(b), \ldots, u^{n-1}(b)$ are linearly independent.

The homogeneous boundary value problem (1), (2) contains only a trivial solution $u(x) \equiv 0$.

Green's function of the boundary value problem (1), (2) is the function G (x, ξ) constructed for any point ξ, a < ξ < b satisfying the following properties :

1. G(x, ξ) is continuous in x for fixed ξ and has continuous derivatives with regard to x upto order $(n-2)$ inclusive for $a \leq x \leq b$.
2. Its $(n-1)$th derivative with regard to x at the point x = ξ has a discontinuity of first kind, the jump being equal to $-\dfrac{1}{[p_0(x)]_{x=\xi}}$, that is,

$$\left\{\frac{\partial^{n-1}}{\partial x^{n-1}} G(x,\xi)\right\}_{x=\xi+0} - \left\{\frac{\partial^{n-1}}{\partial x^{n-1}} G(x,\xi)\right\}_{x=\xi-0} = -\frac{1}{p_0(\xi)} \qquad (3)$$

where $G|_{x=\xi+0}$ defines the limit of G(x, ξ) as $x \to \xi$ from the right and $G|_{x=\xi-0}$ defines the limit of G(x, ξ) as $x \to \xi$ from the left.

3. In each of the intervals [a, ξ) and (ξ, b] the function G(x, ξ), considered as a function of x, is a solution of the equation (1)

$$L(G) = 0 \qquad (4)$$

4. The function G(x, ξ) satisfies the boundary conditions (2)

$$V_k(G) = 0, k = 1, 2, 3,\ldots,n, \qquad (5)$$

If the boundary value problem (1), (2) contains only the trivial solution $u(x) \equiv 0$ then the operator L contains one and only one Green's function G(x, ξ).

Consider $u_1(x), u_2(x),\ldots, u_n(x)$ be linearly independent solutions of the equation L(u) = 0. From the condition 1, the unknown Green's function G(x, ξ) must have the representation on the intervals [a, ξ) and (ξ, b]

$$G(x, \xi) = a_1u_1(x) + a_2u_2(x) + \ldots + a_nu_n(x),\ a \leq x < \xi$$

and $$G(x, \xi) = b_1u_1(x) + b_2u_2(x) + \ldots + b_nu_n(x),\ \xi \leq x < b,$$

where $a_1, a_2,\ldots,a_n, b_1, b_2,\ldots, b_n$ are some functions of ξ.

From the condition 1, the continuity of the function $G(x, \xi)$ and of its first $(n-2)$ derivatives with regard to x at the point $x = \xi$ yields

$$[b_1u_1(\xi) + b_2u_2(\xi) + \ldots + b_nu_n(\xi)] - [a_1u_1(\xi) + a_2u_2(\xi) + \ldots + a_nu_n(\xi)] = 0$$

$$[b_1u_1'(\xi) + b_2u_2'(\xi) + \ldots + b_nu_n'(\xi)] - [a_1u_1'(\xi) + a_2u_2'(\xi) + \ldots + a_nu_n'(\xi)] = 0$$

$$[b_1u_1''(\xi) + b_2u_2''(\xi) + \ldots + b_nu_n''(\xi)] - [a_1u_1''(\xi) + a_2u_2''(\xi) + \ldots + a_nu_n''(\xi)] = 0$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$[b_1u_1^{n-2}(\xi) + b_2u_2^{n-2}(\xi) + \ldots + b_nu_n^{n-2}(\xi)] - [a_1u_1^{n-2}(\xi) + a_2u_2^{n-2}(\xi) + \ldots + a_nu_n^{n-2}(\xi)] = 0$$

Also, $$[b_1u_1^{n-1}(\xi) + b_2u_2^{n-1}(\xi) + \ldots + b_nu_n^{n-1}(\xi)] - [a_1u_1^{n-1}(\xi) + a_2u_2^{n-1}(\xi) + \ldots + a_nu_n^{n-1}(\xi)] = -\frac{1}{p_0(\xi)}$$

Assume $C_k(\xi) = b_k(\xi) - a_k(\xi)$, $k = 1, 2, \ldots, n$; then the system of linear equations in $C_k(\xi)$ are obtained

$$C_1u_1(\xi) + C_2u_2(\xi) + \ldots + C_nu_n(\xi) = 0$$

$$C_1u_1'(\xi) + C_2u_2'(\xi) + \ldots + C_nu_n'(\xi) = 0$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$C_1u_1^{n-2}(\xi) + C_2u_2^{n-2}(\xi) + \ldots + C_nu_n^{n-2}(\xi) = 0$$

$$C_1u_1^{n-1}(\xi) + C_2u_2^{n-1}(\xi) + \ldots + C_nu_n^{n-1}(\xi) = -\frac{1}{p_0(\xi)} \qquad (6)$$

The determinant of the system is equal to the value of the Wronskian $W(u_1, u_2, \ldots, u_n)$ at the point $x = \xi$ and is therefore different from zero.

From the boundary conditions (2), we have

$$V_k(u) = A_k(u) + B_k(u) \qquad (7)$$

where $A_k(u) = \alpha_k u(a) + \alpha^1{}_k u'(a) + \alpha^2{}_k u''(a) + \ldots\ldots.. + \alpha_k^{n-1} u^{n-1}(a)$

$$B_k(u) = \beta_k u(b) + \beta^1{}_k u'(b) + \beta^2{}_k u''(b) + \ldots\ldots.. + \beta_k^{n-1} u^{n-1}(b)$$

Using the condition 4, we have

$V_k(G) = a_1A_k(u_1) + a_2A_k(u_2) + \ldots + a_nA_k(u_n) + \ldots + b_1B_k(u_1) + b_2B_k(u_2) + \ldots + b_nB_k(u_n) = 0$,

where $k = 1, 2, \ldots, n$.

Since $a_k = b_k - c_k$, so we have

$$(b_1 - c_1)A_k(u_1) + (b_2 - c_2)A_k(u_2) + \ldots + (b_n - c_n)A_k(u_n) + b_1B_k(u_1) + b_2B_k(u_2) + \ldots + b_nB_k(u_n) = 0$$

$$\Rightarrow \qquad b_1V_k(u_1) + b_2V_k(u_2) + \ldots + b_nV_k(u_n) = c_1A_k(u_1) + c_2A_k(u_2) + \ldots + c_nA_k(u_n) \qquad (8)$$

which is a linear system in the quantities $b_1, b_2, \ldots, b_n$. The determinant of the system is different from zero, that is,

$$\begin{vmatrix} V_1(u_1) & V_1(u_2) & \cdots & V_1(u_n) \\ V_2(u_1) & V_2(u_2) & \cdots & V_2(u_n) \\ \cdots & \cdots & \cdots & \cdots \\ V_n(u_1) & V_n(u_2) & \cdots & V_n(u_n) \end{vmatrix} \neq 0$$

The system of equations (8) contain a unique solution in $b_1(\xi)$, $b_2(\xi)$, ..., $b_n(\xi)$ and since

$a_k(\xi) = b_k(\xi) - c_k(\xi)$, it follows that the quantities $a_k(\xi)$ are defined uniquely.

I. If the boundary value problem (1), (2) is self – adjoint, then Green's function is symmetric, that is, $G(x, \xi) = G(\xi, x)$. The converse is true as well.

II. If at one of the extremities of an interval [a, b], the coefficient of the derivative vanishes. For example, $p_0(a) = 0$, then the natural boundary condition for the boundedness of the solution x = a is imposed, and at the other extremity the ordinary boundary condition is specified.

3.2.1. Particular case. We shall construct the Green's Function $G(x, \xi)$ for a given number ξ, for the second differential equation

$$L(u) + \phi(x) = 0 \tag{1}$$

where
$$L \equiv \frac{d}{dx}\left(p\,\frac{d}{dx}\right) + q \tag{2}$$

Together with the homogenous boundary conditions of the form

$$\alpha u + \beta \frac{du}{dx} = 0 \tag{3}$$

The Green's function $G(x, \xi)$ constructed for any point $\xi, a < \xi < b$ contains the following properties:

1. $G_1(\xi) = G_2(\xi)$; it follows that the function $G(x, \xi)$ is continuous in x for fixed ξ, in particular, continuous at the point x = ξ.

2. The derivatives of G(which are of finite magnitude) are continuous at every point within the range of x except at x = ξ where it is continuous so that

$$G_2'(\xi) - G_1'(\xi) - \frac{1}{p(\xi)}$$

3. The functions G_1 and G_2 satisfy homogenous conditions at the end points x = a and x = b respectively.

4. The function G_1 and G_2 satisfy the homogenous equations LG = 0 in their defined intervals except at z = ξ, that is, $LG_1 = 0, x < \xi$, $L\,G_2 = 0,\ x > \xi$.

Consider the Green's function $G(x,\xi)$ exists, then the solution of the given differential equation can be transformed to the relation

$$u(x)=\int_a^b G(x,\xi)\,\phi(\xi)\,d\xi \tag{4}$$

Consider two linearly independent solutions of the homogeneous equation $L(u)=0$. *Let* $u=v_1(x)$ and $u=u_2(x)$ be the non-trivial solution of the equation, which satisfy the homogenous conditions at x = a and x = b respectively.

Consider the Green's functions for the problem from the conditions III and IV, in the form

$$G(x,\xi)=\begin{cases} C_1u_1(x), & x<\xi \\ C_2\,u_2(x), & x<\xi \end{cases} \tag{5}$$

where the constant C_1 and C_2 are chosen in a manner that the conditions I and II are fulfilled. Thus, we have

$$C_2u_2(\xi)-C_1u_1(\xi)=0$$

$$C_2u_2'(\xi)-C_1u_1'(\xi)-\frac{1}{p(\xi)} \tag{6}$$

The determinant of the system (6) is the Wronskian $W\left[u_1(\xi),\ u_2(\xi)\right]$ evaluated at the point x = ξ for linearly independent solution $u_1(x)$ and $u_2(x)$, and, hence it is different from zero $W(\xi)\neq 0$

$$W\left[u_1(\xi),u_2(\xi)\right]=\begin{vmatrix} u_1(\xi) & u_2(\xi) \\ u_1'(\xi) & u_2'(\xi) \end{vmatrix}=u_1(\xi)u_2'(\xi)-u_2(\xi)u_1'(\xi) \tag{7}$$

By using Abel's formula, we notice that the expression has the value {C/p(ξ)}, where C is a constant independent of ξ, that is,

$$u_1(\xi)u_2'(\xi)-u_2(\xi)u_1'(\xi)=\frac{C}{p(\xi)} \tag{8}$$

From the system (6), we have

$$C_1=-\frac{1}{C}\,u_2(\xi)\ ,\ C_2=-\frac{1}{C}\,u_1(\xi)$$

Thus the relation (5) reduces to

$$G(x,\xi)=\begin{cases} -\dfrac{1}{C}u_1(x)\,u_2(\xi), & x<\xi \\ -\dfrac{1}{C}u_1(\xi)u_2(x), & x>\xi \end{cases} \tag{9}$$

This result breaks down iff C vanishes, so that u_1 and u_2 are linearly dependent, and hence are each multiples of a certain non-trivia function U(x). In this case, the function u(x) satisfies the equation L(u) = 0 together with the end conditions at x = a, x = b.

Converse. The integral equation

$$u(x) = \int_a^b G(x, \xi)\, \phi(\xi)\, d\xi \qquad (10)$$

where G(x, ξ) are defined by the relation (9), satisfy the differential equation

$$L(u) + \phi(x) = 0 \qquad (11)$$

together with the prescribed boundary condition.

We know that

$$u(x) = -\frac{1}{C}\left[\int_a^x u_1(\xi)\, u_2(x)\, \phi(\xi)\, d\xi + \int_x^b u_1(x)\, u_2(\xi)\, \phi(\xi)\, d\xi\right] \qquad (12)$$

$$u'(x) = -\frac{1}{C}\left[\int_a^x u_2'(x)\, u_1(\xi)\, \phi(\xi)\, d\xi + \int_x^b u_1'(x)\, u_2(\xi)\, \phi(\xi)\, d\xi\right] \qquad (13)$$

$$u''(x) = -\frac{1}{C}\left[\int_a^x u_2''(x)\, u_1(\xi)\, \phi(\xi)\, d\xi + \int_x^b u_1''(x)\, u_2(\xi)\, \phi(\xi)\, d\xi\right]$$

$$-\frac{1}{C}\left[u_2'(x)\, u_1(x) - u_1'(x)\, u_2(x)\right]\phi(x) \qquad (14)$$

Since L(u) $\equiv$ $p(x)u''(x) + p(x)u'(x) + q(x)\, u(x)$

Thus,

$$Lu(x) = -\frac{1}{C}\left[\int_a^x \{Lu_2(x)\}\, u_1(\xi)\, \phi(\xi)\, d\xi + \int_x^b \{Lu_2(x)\}\, u_2(\xi)\, \phi(\xi)\, d\xi\right] - \frac{1}{C}\left[p(x).\frac{C}{p(x)}\phi(x)\right]$$

Again, $u_1(x)$ and $u_2(x)$ satisfy L (u) = 0, hence the first two terms vanish identically.

So, L u(x) = $-\phi(x)$ $\Rightarrow$ L u(x) + $\phi(x)$ = 0

Therefore, a function u(x) satisfying (10) also satisfies the differential equation (11)

Again from (12) and (13), we have

$$u(a) = -\frac{u_1(a)}{C}\int_a^b u_2(\xi)\, \phi(\xi)\, d\xi$$

$$u'(a) = -\frac{u_1'(b)}{C}\int_a^b u_2(\xi)\, \phi(\xi)\, d\xi$$

which shows that the function u defined by (11) satisfies the same homogeneous condition at x = a as the function u_1.

Note. Let $\phi(x) = \lambda r(x) u(x) - f(x)$.

From the differential equation (1), we have

$$Lu(x) + \lambda r(x) u(x) = f(x) \tag{15}$$

The corresponding Fredholm integral equation becomes

$$u(x) = \lambda \int_a^b G(x, \xi)\, r(\xi)\, u(\xi)\, d\xi - \int_a^b G(x, \xi)\, f(\xi)\, d\xi \tag{16}$$

where $G(x, \xi)$ is the Green's function.

From (9), it follows that $G(x, \xi)$ is symmetric but the kernel $K(x, \xi)$ $\{= G(x, \xi) r(\xi)\}$ is not symmetric unless $r(x)$ is a constant.

Consider $\sqrt{\{r(x)\}}u(x) = V(x)$ with the assumption that $r(x)$ is non – negative over (a, b). This equation (16) may be expressed as

$$\frac{V(x)}{\sqrt{r(x)}} = \lambda \int_a^b G(x, \xi) \sqrt{r(\xi)}\, V(\xi)\, d\xi - \int_a^b G(x, \xi)\, f(\xi)\, d\xi$$

or

$$V(x) = \lambda \int_a^b K(x, \xi)\, V(\xi)\, d\xi - \int_a^b K(x, \xi)\, \frac{f(\xi)}{\sqrt{r(\xi)}}\, d\xi, \tag{17}$$

where $K(x, \xi) = \sqrt{\{r(x) r(\xi)\}}\, G(x, \xi)$ and hence possesses the same symmetry as $G(x, \xi)$.

3.2.2. Example. Construct an integral equation corresponding to the boundary value problem.

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (\lambda x^2 - 1)\, u = 0, \tag{1}$$

$$u(0) = 0, \; u(1) = 0 \tag{2}$$

Solution. The differential equation (1) may be written as

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \left(-\frac{1}{x} + \lambda x\right) u = 0.$$

or

$$\left[\frac{d}{dx}\left(x\frac{du}{dx}\right) - \frac{u}{x}\right] + \lambda x u = 0.$$

Comparing with the equation (15), we have

$$p = x, \; q = -\frac{1}{x}, \; r = x \tag{3}$$

The general solution of the homogeneous equation

$L(u) = 0 \quad \Rightarrow \quad \left\{\frac{d}{dx}\left(x\frac{du}{dx}\right) - \frac{u}{x}\right\} = 0$ is given by

$$u(x) = C_1x + C_2\left(\frac{1}{x}\right)$$

Consider $u = u_1(x)$ and $u = u_2(x)$ be the non – trivial solutions of the equation, which satisfy the conditions at x = 0 and x = 1 respectively then

$$u_1(x) = x \qquad \text{and } u_2(x) = \frac{1}{x} - x .$$

The Wronskian of $u_1(x)$ and $u_2(x)$ is given by

$$W[u_1(x), u_2(x)] = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = x\left(-\frac{1}{x^2} - 1\right) - \left(\frac{1}{x} - x\right) = -\frac{2}{x}$$

So, $$u_1(x)u_2'(x) - u_2(x)u_1'(x) = -\frac{2}{x} \Rightarrow C = -2$$

Thus from the relation (19), we have

$$G(x, \xi) = \begin{cases} \dfrac{1}{2}\dfrac{x}{\xi}(1-\xi^2) , & x < \xi , \\ \dfrac{1}{2}\dfrac{\xi}{x}(1-x^2) , & x > \xi , \end{cases} \tag{4}$$

Therefore, from (16), the corresponding Fredholm integral equation becomes

$u(x) = \lambda\int_0^1 G(x , \xi)\, \xi\, u(\xi)\, d\xi$, where the Green's function $G(x, \xi)$ is defined by the relation (4).

3.2.3. Example. Construct Green's function for the homogeneous boundary value problem

$$\frac{d^4u}{dx^4} = 0 \text{ with the conditions } u(0) = u'(0) = 0,\ u(1) = u'(1) = 0.$$

Solution. The differential equation is given by

$$\frac{d^4u}{dx^4} = 0 \tag{1}$$

We notice that the boundary value problem contains only a trivial solution. The fundamental system of solutions for the differential equation (1) is

$$u_1(x) = 1,\ u_2(x) = x,\ u_3(x) = x^2,\ u_4(x) = x^3 \tag{2}$$

Its general solution is of the form

$$u(x) = A + Bx + Cx^2 + Dx^3,$$

where A, B, C, D are arbitrary constants. The boundary conditions give the relations for determining the constants A, B, C, D :

$$u(0) = 0 \quad \Rightarrow \quad A = 0, \quad u'(0) = 0 \quad \Rightarrow \quad B = 0$$

u(1) = 0 $\Rightarrow$ A + B + C + D = 0, $u'(1) = 0$ $\Rightarrow$ B + 2C + 3D = 0

$\Rightarrow$ A = B = C = D = 0.

Thus the boundary value problem has only a zero solution u(x) $\equiv$ 0 and hence we can construct a unique Green's function for it.

<u>*Construction of Green's Function:*</u> **C**onsider the unknown Green's function G(x, ξ) must have the representation on the interval [0, ξ) and (ξ, 1].

$$G(x, \xi) = \begin{cases} a_1.1 + a_2.x + a_3.x^2 + a_4.x^3 &, \quad 0 \le x \le \xi \\ b_1.1 + b_2.x + b_3.x^2 + b_4.x^3 &, \quad \xi \le x \le 1 \end{cases} \tag{3}$$

where a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 are the unknown functions of ξ.

Consider $C_k = b_k(\xi) - a_k(\xi)$, k = 1, 2, 3, 4,… (4)

The system of linear equations for determining the functions $C_k(\xi)$ become

$$C_1 + C_2\xi + C_3\xi^2 + C_4\xi^3 = 0$$

$$C_2 + 2C_3\xi + 3C_4\xi^2 = 0$$

$$2C_3 + 6C_4\xi = 0$$

$$6C_4 = 1$$

$$\Rightarrow \quad C_4(\xi) = \frac{1}{6},\ C_3(\xi) = -\frac{1}{2}\xi,\ C_2(\xi) = \frac{1}{2}\xi^2,\ C_1(\xi) = -\frac{1}{6}\xi^3 \tag{5}$$

From the property 4 of Green's function, it must satisfy the boundary conditions :

$$G(0, \xi) = 0,\ G'_x(0, \xi) = 0$$

$$G(1, \xi) = 0,\ G'_x(1, \xi) = 0$$

The relations reduce to

$$a_1 = 0,\ a_2 = 0$$

$$b_1 + b_2 + b_3 + b_4 = 0$$

$$b_2 + 2b_3 + 3b_4 = 0 \tag{6}$$

From the relation (4), (5) and (6), we have

$$C_1 = b_1(\xi) - a_1(\xi) \Rightarrow b_1(\xi) = -\frac{1}{6}\xi^3$$

or $$C_2 = b_2(\xi) - a_2(\xi) \Rightarrow b_2(\xi) = \frac{1}{2}\xi^2$$

or $b_3 + b_4 = \frac{1}{6}\ \xi^3,\ \frac{1}{2}\ \xi^2,\ 2b_3 + 3b_4 = -\ \frac{1}{2}\ \xi^2$

$\Rightarrow$ $b_4(\xi) = \frac{1}{2}\ \xi^2 - \frac{1}{3}\ \xi^3$ and $b_3(\xi) = \frac{1}{2}\ \xi^3 - \xi^2$

or $C_3(\xi) = b_3(\xi) - a_3(\xi)$

$\Rightarrow$ $a_3(\xi) = b_3(\xi) - C_3(\xi) = \frac{1}{2}\ \xi^3 - \xi^2 + \frac{1}{2}\ \xi$

and $C_4(\xi) = b_4(\xi) - a_4(\xi)$

$\Rightarrow$ $a_4(\xi) = b_4(\xi) - C_4(\xi) = \frac{1}{2}\ \xi^2 - \frac{1}{3}\ \xi^3 - \frac{1}{6}$

Substituting the value of the constants a_1, a_2, a_3, a_4, b_1, b_2, C_3, C_4 in the relation (3), the Green's function $G(x, \xi)$ is obtained as

$$G(x,\xi) = \begin{cases} \left(\frac{1}{2}\xi - \xi^2 + \frac{1}{2}\xi^3\right)x^2 - \left(\frac{1}{6} - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3\right)x^3, & 0 \le x \le \xi \\ -\frac{1}{6}\xi^3 + \frac{1}{2}\xi^2 x + \left(\frac{1}{2}\xi^3 - \xi^2\right)x^2 + \left(\frac{1}{2}\xi^2 - \frac{1}{3}\xi^3\right)x^3, & \xi \le x \le 1 \end{cases}$$

The expression $G(x, \xi)$ may be transformed to

$$G(x, \xi) = \left(\frac{1}{2}x - x^2 + \frac{1}{2}x^3\right)\xi^2 - \left(\frac{1}{6} - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\xi^3, \quad \xi \le x \le 1$$

$\Rightarrow$ $G(x, \xi) = G(\xi, x)$, that is, Green's function is symmetric.

3.2.4. Example. Construct Green's function for the equation $x\frac{d^2u}{dx^2} + \frac{du}{dx} = 0$ with the conditions u(x) is bounded as $x \to 0$, $u(1) = \mu u'(1)$, $\mu \neq 0$.

Solution. The differential equation is given by $x\frac{d^2u}{dx^2} + \frac{du}{dx} = 0$ (1)

or $\left(\frac{d^2u/dx^2}{du/dx}\right)dx = -\frac{1}{x}dx$

or $\log\frac{du}{dx} = -\log x + \log A$

or $\frac{du}{dx} = \frac{A}{x}$

or $u(x) = A\log x + B$ (2)

The conditions u(x) is bounded as $x \to 0$ and $u(1) = \mu u'(1)$, $\mu \neq 0$ has only a trivial solution $u(x) \equiv 0$, hence we can construct a (unique) Green's function $G(x, \xi)$

Consider the function $G(x, \xi)$ as:

$$G(x, \xi) = \begin{cases} a_1 + a_2 \log x, & 0 < x \leq \xi \\ b_1 + b_2 \log x, & \xi \leq x \leq 1 \end{cases} \tag{3}$$

where a_1, a_2, b_1, b_2 are unknown functions of ξ.

Consider $C_k = b_k(\xi) - a_k(\xi)$, k = 1, 2,…

From the continuity of $G(x, \xi)$ for $x = \xi$, we obtain

$$b_1 + b_2 \log \xi - a_1 - a_2 \log \xi = 0$$

and the jump $G'_x(x, \xi)$ at the point $x = \xi$ is equal to $\frac{1}{\xi}$ so that

$$b_2 . \frac{1}{\xi} - a_2 . \frac{1}{\xi} = -\frac{1}{\xi}$$

Putting $\quad C_1 = b_1 - a_1,\ C_2 = b_2 - a_2 \quad$ (4)

$\Rightarrow \quad C_1 + C_2 \log \xi = 0,\ C_2 = -1.$

Hence $\quad C_1 = \log \xi \quad$ and $\quad C_2 = -1 \quad$ (5)

The boundedness of the function $G(x, \xi)$ as $x \to 0$ gives $a_2 = 0$

Also, $\quad G(x, \xi) = \mu G'_x(x, \xi),\ b_1 = \mu b_2$

$\Rightarrow \quad a_1 = -(\mu + \log \xi),\ a_2 = 0,\ b_1 = -1,\ b_2 = -\mu$

Substituting the value of the constants a_1, a_2, b_1, b_2 in the relation (3), the Green's function is obtained as

$$G(x, \xi) = \begin{cases} -(\mu + \log \xi), & 0 < x \leq \xi \\ -(1 + \mu \log x), & \xi \leq x \leq 1 \end{cases}.$$

3.2.5. Exercise.

1. Construct the Green's function for the boundary value problem $u''(x) + \mu^2 u = 0$ with the conditions $u(0) = u(1) = 0$.

Answer. $$G(x, \xi) = \begin{cases} \dfrac{\sin \mu(\xi - 1) \sin \mu x}{\mu \sin \mu}, & 0 \leq x \leq \xi \\ \dfrac{\sin \mu \xi \sin \mu(x - 1)}{\mu \sin \mu}, & \xi < x \leq 1 \end{cases}$$

2. Find the Green's function for the boundary value problem $\frac{d^2u}{dx^2} - u(x) = 0$ with the conditions u(0) = u(1) = 0.

Answer. $G(x, \xi) = \begin{cases} \dfrac{\sinh x \sinh(\xi - 1)}{\sinh 1} & , \quad 0 \le x \le \xi \\ \dfrac{\sinh \xi \sinh(x - 1)}{\sinh 1} & , \quad \xi \le x \le 1 \end{cases}$.

3.2.6. Article. If u(x) has continuous first and second derivatives, and satisfies the boundary value problem $\frac{d^2u}{dx^2} + \lambda u = 0$ with u(0) = u(l) = 0 then u(x) is continuous and satisfies the homogeneous linear integral equation $u(x) = \lambda \int_0^1 G(x, \xi)\, u(\xi)\, d\xi$.

Solution : The differential equation may be written as

$$\frac{d^2u}{dx^2} + \lambda u = 0 \Rightarrow \frac{d^2u}{dx^2} = -\lambda u \tag{1}$$

By integrating with regard to x over the interval (0, x) two times, we obtain

$$\frac{du}{dx} = -\lambda \int_0^x u(\xi)\, d\xi + C$$

or
$$u(x) = -\lambda \int_0^x (x - \xi)\, u(\xi)\, d\xi + C_x + D \tag{2}$$

where C and D are the integration constants, to be determined by the boundary conditions.

$$u(0) = 0 \quad \Rightarrow \quad D = 0$$

$$u(l) = 0 \quad \Rightarrow \quad -\lambda \int_0^l (l - \xi)\, u(\xi)\, d\xi + Cl = 0$$

$$\Rightarrow \quad C = \frac{\lambda}{l} \int_0^l (l - \xi)\, u(\xi)\, d\xi$$

Substituting the value of the constants C and D in (2), we have

$$u(x) = -\lambda \int_0^x (x - \xi)\, u(\xi)\, d\xi + \frac{\lambda}{l} \int_0^l x(l - \xi)\, u(\xi)\, d\xi$$

or
$$u(x) = -\lambda \int_0^x (x - \xi)\, u(\xi)\, d\xi + \frac{\lambda}{l} \int_0^x x(l - \xi)\, u(\xi)\, d\xi + \frac{\lambda}{l} \int_x^l x(l - \xi)\, u(\xi)\, d\xi$$

or
$$u(x) = \lambda \int_0^x \frac{\xi}{l}(l - x)\, u(\xi)\, d\xi + \lambda \int_x^l \frac{x}{l}(l - \xi)\, u(\xi)\, d\xi$$

or
$$u(x) = \lambda \int_0^l G(x, \xi)\, u(\xi)\, d\xi$$

where $\qquad G(x, \xi) = \begin{cases} \dfrac{\xi}{l}(l-x) & , x > \xi \\ \dfrac{x}{l}(l-\xi) & , x < \xi \end{cases}$.

3.2.7. Exercise.

1. Construct the Green's function for the boundary value problem $\dfrac{d^2u}{dx^2} + \mu^2 u = 0$ with the conditions u(0) = u(l) = 0.

   Answer. $G(x, \xi) = \begin{cases} a_1 \cos \mu x + a_2 \sin \mu x = -\dfrac{\sin \mu(\xi - l) \sin \mu x}{\mu \sin \mu l} & , \quad 0 \le x < \xi \\ b_1 \cos \mu x + b_2 \sin \mu x = -\dfrac{\sin \mu \xi \sin \mu(x-l)}{\mu \sin \mu l} & , \quad \xi < x \le l \end{cases}$

2. Construct the Green's function for the boundary value problem $\dfrac{d^2u}{dx^2}$ = 0 with the conditions u(0) = $u'(1)$ and $u'(0)$ = u(1).

   Answer. $G(x, \xi) = \begin{cases} (-\xi+2)x + (-\xi+1) & , \quad 0 \le x < \xi \\ (-\xi+1)x + 1 & , \quad \xi < x \le 1 \end{cases}$

3. Construct the Green's function for the boundary value problem $\dfrac{d^3u}{dx^3}$ = 0 with the boundary conditions u(0) = $u'(1)$ = 0 and $u'(0)$ = u(1).

   Answer. $G(x, \xi) = \begin{cases} \dfrac{1}{2} x(\xi-1)[x - x\xi + 2\xi] & 0 \le x < \xi \\ \dfrac{1}{2}\xi[x(2-x)(\xi-2) + \xi] & \xi < x \le 1 \end{cases}$

4. Construct the Green's function for the boundary value problem $x^2 \dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} - u$ = 0 with u(x) is bounded as $x \to 0$ and u(1) = 0.

   Answer. $G(x, \xi) = \begin{cases} \dfrac{1}{2} x\left(\dfrac{1}{\xi^2} - 1\right) & , \quad 0 \le x < \xi \\ \dfrac{1}{2}\left(\dfrac{1}{x} - x\right) & , \quad \xi < x \le 1 \end{cases}$

5. Construct the Green's function for the boundary value problem $\dfrac{d^2u}{dx^2} - u$ = 0 with the conditions u(0) = $u'(0)$ and u(l) + λ $u'(l)$ = 0.

Answer. G(x, ξ) = $\begin{cases} -\frac{1}{2}\left(\frac{1-\lambda}{1+\lambda}\right)e^{x+\xi-2l}+\frac{1}{2}e^{x-\xi} & , \ 0\le x<\xi \\ -\frac{1}{2}\left(\frac{1-\lambda}{1+\lambda}\right)e^{x+\xi-2l}+\frac{1}{2}e^{\xi-x} & , \ \xi<x\le 1 \end{cases}$, where $|\lambda|\neq 1$.

6. Using Green's function, solve the boundary value problem u''(x) $-$u(x) = x with boundary conditions u(0) = u(1) = 0.

Answer. Here, G(x, ξ) = $\begin{cases} -\frac{\sinh x \sinh(\xi-1)}{\sinh 1} & , 0\le x<\xi \\ -\frac{\sinh \xi \sinh(x-1)}{\sinh 1} & , \xi < x\le 1 \end{cases}$ and the solution of the given boundary value problem is given by u(x) = $\int_0^1 G(x\,,\,\xi)\xi\, d\xi$, so u(x) = $\frac{\sinh x}{\sinh 1}-x$.

7. Using Green's function, solve the boundary value problem $\frac{d^2u}{dx^2}+u = x$ with the boundary conditions u(0) = 0 and u($\pi/2$) = 0.

Answer. Here, G(x, ξ) = $\begin{cases} \cos\xi \sin x & , \ 0\le x<\xi \\ \sin\xi\cos x & , \ \xi<x\le \pi/2 \end{cases}$ and u(x) = $\int_0^{\pi/2} G(x\,,\,\xi)\,\xi\, d\xi$, implies

u(x) = x $-$ $\frac{\pi}{2}$sin x.

8. Solve the boundary value problem using Green's function

$$\frac{d^2u}{dx^2}+u = x^2 ;\ u(0) = u(\pi/2) = 0.$$

Answer. u(x) = $-\left[2\cos x+\sin x\left(2-\frac{\pi^2}{4}\right)+x^2-2\right]$.

3.3. Construction of Green's function when the boundary value problem contains a parameter.

Consider a differential equation of order n

$$L(u) - \lambda h = h(x) \tag{1}$$

with $$V_k(u) = 0,\ k = 1, 2, 3,\ldots,n \tag{2}$$

where $$L(u)\equiv p_0(x)u^n(x)+p_1(x)u^{n-1}(x)+\ldots+p_n(x)u(x) \tag{3}$$

and $V_k(u)\equiv \alpha_k u(a)+\alpha_k^1 u'(a)+\ldots+\alpha_k^{n-1}u^{n-1}(a)+\ldots+\beta_k u(b)+\beta_k^1 u'(b)+\ldots+\beta_k^{n-1}u^{n-1}(b)+\ldots$ (4)

where the linear forms V_1, V_2,…, V_n in u(a), u'(a), …, u^{n-1}(a), u(b), u'(b), …, u^{n-1}(b) are linearly independent, h(x) is a given continuous function of x, λ is some non–zero numerical parameter.

For $h(x) \equiv 0$, the equation (1) reduces to homogeneous boundary value problem

$$L(u) = \lambda u ,$$
$$V_k(u) = 0,\ k = 1, 2, 3, \ldots, n \tag{5}$$

Those values of λ for which the boundary value problem (5) has non trivial solutions u(x) are called the eigenvalues. The non–trivial solutions are called the associated eigen functions.

If the boundary value problem

$$L(u) = 0,$$
$$V_k(u) = 0,\ k = 1, 2, \ldots, n \tag{6}$$

contains the Green's function $G(x, \xi)$, then the boundary value problem (1) and (2) is equivalent to the Fredholm integral equation

$$u(x) = \lambda\int_a^b G(x , \xi)\ u(\xi)\ d\xi + f(x) \tag{7}$$

where $$f(x) = \int_a^b G(x , \xi)\ h(\xi)\ d\xi \tag{8}$$

In particular, the homogeneous boundary value problem (5) is equivalent to the homogeneous integral equation

$$u(x) = \lambda\int_a^b G(x , \xi)\ u(\xi)\ d\xi \tag{9}$$

Since $G(x, \xi)$ is a continuous kernel, therefore the Fredholm homogeneous integral equation of second kind (9) can have at most a countable number of eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ which do not have a finite limit point. For all values of λ different from the eigen values, the non-homogeneous equation (7) has a solution for any continuous function f(x). Thus the solution is given by

$$u(x) = \lambda\int_a^b R(x , \xi;\lambda)\ f(\xi)\ d\xi + f(\xi) \tag{10}$$

where $R(x, \xi ; \lambda)$ is the resolvent kernel of the kernel $G(x, \xi)$. The function $R(x, \xi ; \lambda)$ is a meromorphic function of λ for any fixed values of x and ξ in [a, b]. The eigen values of the homogeneous integral equation (9) may by the pole of this function.

3.3.1. Example. Reduce the boundary value problem $\frac{d^2u}{dx^2} + \lambda u = x$, $u(0) = u(\pi/2) = 0$, to an integral equation using Green's function.

Solution. Consider the associated boundary value problem

$$\frac{d^2u}{dx^2} = 0 \tag{1}$$

whose general solution is given by $u(x) = Ax + B$

The boundary conditions u(0) = 0, u($\pi/2$) = 0 yields only the trivial solution u(x) $\equiv$ 0. Therefore, the Green's function G(x, ξ) exists for the associated boundary value problem

$$G(x, \xi) = \begin{cases} a_1 x + a_2 \ , & 0 \le x < \xi \\ b_1 x + b_2 \ , & \xi < x \le \pi/2 \end{cases} \tag{2}$$

The Green's function G(x, ξ) must satisfy the following properties :

(I) The function G(x, ξ) is continuous at x = ξ, that is,

$$b_1 \xi + b_2 = a_1 \xi + a_2$$

$$\Rightarrow \quad (b_1 - a_1)\ \xi + (b_2 - a_2) = 0 \tag{3}$$

(II) The derivative G(x, ξ) has a discontinuity of magnitude $-\left\{\dfrac{1}{p_0(\xi)}\right\}$ at the point x = ξ,

that is, $$\left(\frac{\partial G}{\partial x}\right)_{x\,=\,\xi+0} - \left(\frac{\partial G}{\partial x}\right)_{x\,=\,\xi-0} = -1 \ \Rightarrow\ b_1 - a_1 = -1 \tag{4}$$

(III) The function G(x, ξ) must satisfy the boundary conditions

$$G(0, \xi) = 0 \qquad \Rightarrow \qquad a_2 = 0 \tag{5}$$

$$G(\pi/2, \xi) = 0 \qquad \Rightarrow \qquad b_1\left(\frac{\pi}{2}\right) + b_2 = 0 \tag{6}$$

Solving the equations (3), (4), (5) and (6), we have

$$a_1 = 1 - \frac{2\xi}{\pi},\ a_2 = 0,\ b_2 = \xi,\ b_1 = -\frac{2\xi}{\pi}.$$

Substituting the value of the constants in (2), the required Green's function G(x, ξ) is obtained

$$G(x, \xi) = \begin{cases} \left(1 - \dfrac{2\xi}{\pi}\right) x \ , & 0 \le x < \xi \\ \left(1 - \dfrac{2x}{\pi}\right) \xi \ , & \xi < x \le \pi/2 \end{cases} \tag{7}$$

Consider the Green's function G(x, ξ) given by the relation (7) as the kernel of the integral equation, we obtain the integral equation for u(x) :

$$u(x) = f(x) - \lambda \int_0^{\pi/2} G(x, \xi)\ u(\xi)\, d\xi \text{, where } f(x) = \int_0^{\pi/2} G(x, \xi)\ \xi\ d\xi$$

or $$f(x) = \int_0^x \left(1 - \frac{2x}{\pi}\right) \xi^2\ d\xi + \int_x^{\pi/2} \left(1 - \frac{2\xi}{\pi}\right) x \xi d\xi$$

or $\quad f(x) = \frac{1}{3}\left(1-\frac{2x}{\pi}\right)x^3 + x\left(\frac{1}{2}\xi^2 - \frac{2}{3\pi}\xi^3\right)_x^{\pi/2}$

or $\quad f(x) = \frac{1}{3}x^3 - \frac{2}{3\pi}x^4 + \frac{\pi^2 x}{24} - \frac{1}{2}x^3 + \frac{2}{3\pi}x^4$

or $\quad f(x) = \frac{\pi^2}{24}x - \frac{x^3}{6}$

Thus, the given boundary value problem has been reduced to an integral equation

$$u(x) + \lambda\int_0^{\pi/2} G(x\,,\,\xi)\; u(\xi)\,d\xi = \frac{\pi^2}{24}x - \frac{1}{6}x^3.$$

3.3.2. Exercise.

1. Reduce the boundary value problem $\frac{d^2u}{dx^2} + xu = 1$, $u(0) = u(1) = 0$ to an integral equation.

Answer. $G(x,\ \xi) = \int_0^x \xi(1-x)\,d\xi + \int_x^1 x(1-\xi)\,d\xi = \frac{1}{2}x(1-x)$, and the required integral equation is $u(x) = \int_0^1 G(x\,,\,\xi)\;\xi u(\xi)\,d\xi - \frac{1}{2}x(1-x)$

2. Reduce the boundary value problem to an integral equation

$$\frac{d^2u}{dx^2} = \lambda u + 1,\ u(0) = u'(0) = 0,\ u''(1) = u'''(1) = 0$$

Answer. $u(x) = \lambda\int_0^1 G(x\,,\,\xi)\;u(\xi)\,d\xi + f(x)$, where $f(x) = \frac{1}{24}x^2(x^2-4x+6)$

3. Reduce the boundary value problem $\frac{d^2u}{dx^2} + \frac{\pi^2}{4}u = \lambda u + \cos\frac{\pi x}{2}$, with $u(-1) = u(1)$ and $u'(-1) = u'(1)$ to an integral equation.

Answer. Here, $G(x,\ \xi) = \begin{cases} \frac{1}{\pi}\sin\frac{\pi}{2}(x-\xi) \quad , & -1 \le x < \xi \\ \frac{1}{\pi}\sin\frac{\pi}{2}(\xi-x) \quad , & \xi < x \le 1 \end{cases}$

and $u(x) = \lambda\int_{-1}^1 G(x\,,\,\xi)\;u(\xi)\,d\xi - \left(\frac{x}{\pi}\sin\frac{\pi x}{2} + \frac{2}{\pi^2}\cos\frac{\pi x}{2}\right)$.

4. Reduce the following boundary value problems to integral equations.

(a) $\quad u'' + \lambda u = 2x + 1$, $u(0) = u'(1)$, $\quad u'(0) = u(1)$

(b) $\quad u'' + \lambda u = e^x$, $\quad u(0) = u''(0)$, $u(1) = u'(1)$.

Answer. (a) Here, G(x, ξ) = $\begin{cases} -\{(\xi-2)x+(\xi-1)\} & , \quad 0 \le x < \xi \\ -\{(\xi-1)x-1\} & , \quad \xi < x \le 1 \end{cases}$ and the boundary value problem reduces to the integral equation

$$u(x) = -\lambda\int_0^1 G(x\,,\,\xi)\,u(\xi)\,d\xi - \frac{1}{6}(2x^3+3x^2-17x-5)\,.$$

(b) Here, G(x, ξ) = $\begin{cases} -(1+x)\xi & , \quad 0 \le x < \xi \\ -(1+\xi)x & , \quad \xi < x \le 1 \end{cases}$ and the boundary value problem reduces to

$$u(x) = -\lambda\int_0^1 G(x\,,\,\xi)\,u(\xi)\,d\xi - e^x\,.$$

5. Reduce the Bessel's differential equation $x^2\dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} + (\lambda x^2 - 1)u = 0$ with the conditions u(0) = 0, u(1) = 0 into an integral equation.

Answer.: The standard equation of Bessel's equation is given by

Here, G(x, ξ) = $\begin{cases} \dfrac{x}{2\xi}(1-\xi^2)\,, & 0 \le x < \xi \\ \dfrac{\xi}{2x}(1-x^2)\,, & \xi < x \le 1 \end{cases}$ and the integral equation can be obtained as

$$u(x) = \lambda\int_0^1 G(x\,,\,\xi)\,r(\xi)\,u(\xi)\,d\xi\,.$$

6. Determine the Green's function G(x, ξ) for the differential equation $\left[\dfrac{d}{dx}\left(x\dfrac{d}{dx}\right) - \dfrac{n^2}{x}\right]u = 0$ with the conditions u(0) = 0 and u(1) = 0.

Answer. G(x, ξ) = $\begin{cases} \dfrac{x^n}{2n\xi^n}(1-\xi^{2n})\,, & x < \xi \\ \dfrac{\xi^n}{2nx^n}(1-x^{2n})\,, & x > \xi. \end{cases}$

3.4. Non–homogeneous ordinary Equation. The boundary value problem associated with a non – homogenous ordinary differential equation of second order is

$$Ly \equiv A_0(x)\frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_2(x)\,y = f(x),\ a < x < b \qquad (1)$$

with boundary conditions $\left.\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0 \end{aligned}\right\}$ (2)

where α_1, α_2, β_1 and β_2 are constants.

3.4.1. Self–Adjoint Operator. The operator L is said to be self – adjoint if for any two functions say u(x) and v(x) operated on L, the expression (vLu − uLv) dx is an exact differential that is, there exist a function g such that dg = (vLu − uLv) dx.

3.4.2. Green's Function Method. Green's function method is an important method to solve B.V.P. associated with non–homogeneous ordinary or partial differential equation . Here we shall show that a B.V.P. will be reduced to a Fredholm integral equation whose kernel is Green's function. We shall be using a special type of B.V.P. namely Sturm – Liouville's problem.

3.4.3. Theorem. Show that the differential operator L of the Sturm – Liouville's Boundary value problem (S.L.B.V.P.)

$$Ly = \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + \left[q(x)+\lambda p(x)\right] y(x) = 0 \qquad (1)$$

with
$$\left.\begin{aligned}\alpha_1 y(a)+\alpha_2 y'(a) &= 0\\ \beta_1 y(b)+\beta_2 y'(b) &= 0\end{aligned}\right\} \qquad (2)$$

where α, β, α_2 and β_2 are constants is self adjoint.

Proof. Let u and v be two solutions of the given S.L.B.V.P. then

$$Lu = \frac{d}{dx}\left[r(x)\frac{du}{dx}\right] + \left[g(x)+\lambda p(x)\right] u(x) = 0$$

and
$$Lv = \frac{d}{dx}\left[r(x)\frac{dv}{dx}\right] + \left[q(x)+\lambda p(x)\right] v(x) = 0$$

So,

$$vLu - uLv = v\frac{d}{dx}\left[r(x)\frac{du}{dx}\right] + \left[q(x)+\lambda p(x)\right] u(x)v - \left[u\frac{d}{dx}\left[r(x)\frac{dv}{dx}\right] + \left[q(x)+\lambda p(x)\right]v(x)u\right]$$

$$= v\frac{d}{dx}\left[r(x)\frac{du}{dx}\right] - u\frac{d}{dx}\left[r(x)\frac{dv}{dx}\right]$$

$$= \left[v\frac{d}{dx}\left(r(x)\frac{du}{dx}\right) + \left(r(x)\frac{du}{dx}\right)\frac{dv}{dx}\right] - \left[u\frac{d}{dx}\left(r(x)\frac{dv}{dx}\right) + \left(r(x)\frac{dv}{dx}\right)\frac{du}{dx}\right]$$

$$= \frac{d}{dx}\left[r(x)\,v(x)\frac{du}{dx}\right] - \frac{d}{dx}\left[r(x)\,u(x)\frac{dv}{dx}\right]$$

$$= \frac{d}{dx}\left[r(x)\,v(x)\frac{du}{dx} - r(x)\,u(x)\frac{dv}{dx}\right] = \frac{d}{dx}\left[r(x)\left(v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}\right)\right] = \frac{dg}{dx}$$

where $g = r(x)\left(v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}\right)$. Then, (vLu − uLv) dx = dg

Hence operator in equation (1) is self – adjoint.

3.4.4. Construction of Green's function by variation of parameter method.

Consider the non – homogeneous differential equation

$$Lu = \frac{d}{dx}\left[r(x)\ \frac{du}{dx}\right] + \left[q(x) + \lambda p(x)\right]\ u(x)\ = f(x) \tag{1}$$

subject to boundary condition:

$$\left.\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a)\ &=\ 0 \\ \beta_1 u(b) + \beta_2 u'(b)\ &=\ 0 \end{aligned}\right\} \tag{*}$$

Construct the Green's function and show that

$$u(x) = -\int_a^b G(x\ ,\ \xi)\ \ f(\xi)\ d\xi \tag{**}$$

where $G(x,\ \xi)$ is the Green's function defined above.

Solution. Let $v_1(x)$ and $v_2(x)$ be two linearly independent solution of the homogeneous differential equation.

$$Lu = \frac{d}{dx}\left[r(x)\frac{du}{dx}\right] +\ \left[q(x) + \lambda p(x)\right]\ u(x) = 0 \tag{2}$$

Then the general solution of (2) by the method of variation of parameters is

$$u(x) = a_1(x)\ v_1(x) + a_2(x)\ v_2(x) \tag{3}$$

where the unknown variables $a_1(x)$ and $a_2(x)$ are to be determined. We assume that neither the solution $v_1(x)$ nor $v_2(x)$ satisfy both the boundary conditions at x = a and x = b but the general solution u(x) satisfies these conditions.

Now, we differentiate (3) w.r.t. x and obtain

$$u'(x)\ =\ a_1'v_1 + a_1v_1' + a_2'v_2 + a_2v_2' \tag{4}$$

Let us equate to zero the terms involving derivatives of parameter, that is,

$$a_1'(x)v_1(x) + a_2'(x)v_2(x)\ = 0 \tag{5}$$

which yields

$$u'(x)\ =\ a_1(x)\ v_1'(x) + a_2(x)\ v_2'(x) \tag{6}$$

Putting the values of u(x) and $u'(x)$ from (3) and (6) respectively in equation (1), we obtain

$$Lu = \frac{d}{dx}\ \left[r(x)\ \left(a_1v_1' + a_2v_2'\right)\right] + \left[q(x) + \lambda p(x)\right](a_1v_1 + a_2v_2) = f(x)$$

or $a_1\left[\frac{d}{dx}(r\,v_1') + v_1(q+\lambda p)\right] + a_2\frac{d}{dx}\left[(r\,v_2') + v_2(q+\lambda p)\right] + \left(a_1'v_1' + a_2'v_2'\right)\,r(x) = f(x)$ (7)

Since v_1 and v_2 are solutions of homogeneous equation (2), so by (7), we get

$$\left(a_1'v_1' + a_2'v_2'\right)r(x) = f(x)$$

$$\Rightarrow \quad a_1'(x)\,v_1'(x) + a_2'(x)\,v_2'(x) = \frac{f(x)}{r(x)} \tag{8}$$

Equations (5) and equation (8) can be solved to get

$$a_1'(x) = \frac{f(x)\,v_2(x)}{r(x)\left[v_2v_1' - v_1v_2'\right]} \quad \text{and} \quad a_2'(x) = \frac{-f(x)\,v_1(x)}{r(x)\left[v_2v_1' - v_1v_2'\right]} \tag{9}$$

Now the operator L is exact and we have proved that

$$v_2\,L\,v_1 - v_1\,L\,v_2 = \frac{d}{dx}\left[r(x)\,\left(v_2v_1' - v_1v_2'\right)\right] \tag{10}$$

Since v_1 and v_2 are solutions of Sturm – Liouville homogeneous differential equation so that $Lv_1 = 0$ and $Lv_2 = 0$ and thus equation (10) gives

$$\frac{d}{dx}\left[r(x)\,\left(v_2v_1' - v_1v_2'\right)\right] = 0$$

$$\Rightarrow \quad r(x)\,\left(v_2v_1' - v_1v_2'\right) = \text{constant} = -\beta \text{ (say)} \tag{11}$$

Thus, equation (9) becomes

$$\left.\begin{array}{l} a_1'(x) = \dfrac{-f(x)\,v_2(x)}{\beta} \quad \text{and} \quad a_2'(x) = \dfrac{f(x)\,v_1(x)}{\beta} \end{array}\right] \tag{12}$$

Integrating (12), we get

$$a_1(x) = -\frac{1}{\beta}\int_{c_1}^{x} f(\xi)\,v_2(\xi)\,d\xi \tag{13}$$

and $$a_2(x) = \frac{1}{\beta}\int_{c_2}^{x} f(\xi)\,v_1(\xi)\,d\xi \tag{14}$$

where c_1 and c_2 are arbitrary constants to be determined from the boundary condition on $a_1(x)$ and $a_2(x)$. These conditions are to be imposed in accordance with our earlier assumption that $v_1(x)$ and $v_2(x)$ does not satisfy boundary conditions but the final solution u(x) satisfies boundary conditions in equation (*). So, that

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0 \tag{15}$$

$$\beta_1 u(b) + \beta_2 u'(b) = 0 \tag{16}$$

Using (3) and (6) in equation (15), we obtain

$$0 = \alpha_1 u(a) + \alpha_2 u'(a)$$

$$= \alpha_1\left[a_1(a)v_1(a) + a_2(a)v_2(a)\right] + \alpha_2\left[a_1(a)v_1'(a) + a_2(a)v_2'(a)\right]$$

$$= a_1(a)\left[\alpha_1 v_1(a) + \alpha_2 v_1'(a)\right] + a_2(a)\left[\alpha_1 v_2(a) + \alpha_2 v_2'(a)\right]$$

Let us now assume that $v_2(x)$ satisfies first boundary condition of (*) but $v_1(x)$ does not satisfy it, then

$$\alpha_1 v_2(a) + \alpha_2 v_2'(a) = 0$$
$$\alpha_1 v_1(a) + \alpha_2 v_2'(a) \neq 0$$

so that $\quad a_1(a)\ \left[\alpha_1 v_1(a) + \alpha_2 v_1'(a)\right] = 0 \Rightarrow a_1(a) = 0$

Using this condition in (13), we get

$$0 = a_1(a) = -\frac{1}{\beta}\int_{c_1}^{a} f(\xi)\, v_2(\xi)\ d\xi \text{ which is satisfied when } c_1 = a$$

Thus, the solution in (13) is :

$$a_1(x) = -\frac{1}{\beta}\int_{a}^{x} f(\xi)\, v_2(\xi)\ d\xi \tag{17}$$

Similarly, using (3) and (6) in (16), we obtain $c_2 = b$ and the solution in (14) is :

$$a_2(x) = \frac{1}{\beta}\int_{b}^{x} f(\xi)\, v_1(\xi)\ d\xi = -\frac{1}{\beta}\int_{x}^{b} f(\xi)\, v_1(\xi)\ d\xi \tag{18}$$

The final solution of the non – homogeneous B.V.P. is

$$u(x) = a_1(x)\, v_1(x) + a_2(x)\, v_2(x)$$

$$= -\frac{1}{\beta}\ v_1(x)\int_{a}^{x} f(\xi)\, v_2(\xi)\, d\xi - \frac{1}{\beta}\ v_2(x)\int_{x}^{b} f(\xi)\, v_1(\xi)\, d\xi$$

$$= -\int_{a}^{x} \frac{v_1(x)\, v_2(\xi)}{\beta}\ f(\xi)\, d\xi - \int_{x}^{b} \frac{v_2(x)\, v_1(\xi)}{\beta}\ f(\xi)\, d\xi = -\int_{a}^{b} G(x, \xi)\, f(\xi)\ d\xi$$

where $G(x, \xi) = \begin{cases} \dfrac{1}{\beta} v_1(x)\, v_2(\xi) & \xi \leq x \leq b \\[2ex] \dfrac{1}{\beta} v_2(x)\, v_1(\xi) & a \leq x \leq \xi \end{cases}$

3.5. Basic Properties of Green’s Function.

3.5.1. Theorem. The Green function $G(x, \xi)$ is symmetric in x and ξ, that is, $G(x, \xi) = G(\xi, x)$.

Proof. Interchanging x and ξ in G(x, ξ) defined above :

$$G(x, \xi) = \begin{cases} \frac{1}{\beta} v_1(\xi)\, v_2(x) & x \le \xi \\ \frac{1}{\beta} v_1(x)\, v_2(\xi) & \xi \le x \end{cases} = G(\xi, x).$$

3.5.2. Theorem. The function G(x, ξ) satisfies the boundary conditions given in equation (*).

Proof. Consider

$$\alpha_1 G(a, \xi) + \alpha_2 G'(a, \xi) = \alpha_1 \left[\frac{v_2(a) v_1(\xi)}{\beta}\right] + \alpha_2 \left[\frac{v_2'(a) v_1(\xi)}{\beta}\right]$$

$$= \frac{1}{\beta}\left[\alpha_1 v_2(a) + \alpha_2 v_2'(a)\right] v_1(\xi)$$

$$= \frac{1}{\beta}[0] v_1(\xi) = 0 \quad x \le \xi \le b$$

Again, $\beta_1 G(b, \xi) + \beta_2 G'(b, \xi) = \beta_1 \left[\frac{v_1(b) v_2(\xi)}{\beta}\right] + \beta_2 \left[\frac{v_1'(b) v_2(\xi)}{\beta}\right]$

$$= \frac{1}{\beta}\left[\beta_1 v_1(b) + \beta_2 v_1'(b)\right] v_2(\xi)$$

$$= \frac{1}{\beta}[0] v_2(\xi) = 0 \quad , a \le \xi \le x.$$

3.5.3. Theorem. The function G(x, ξ) is continuous in [a, b]

Proof. Clearly, G(x, ξ) is continuous at every point of [a, b] except possibly at x = ξ. By definition of G(x, ξ), it can be observed that both branches have same value at x = ξ given by $\frac{1}{\beta}\left[v_1(\xi)\, v_2(\xi)\right]$.

Hence G(x, ξ) is continuous in [a, b].

3.5.4. Theorem. $\frac{\partial G}{\partial x}$ has a jump discontinuity at x = ξ, given by

$$\left.\frac{\partial G}{\partial x}\right|_{x = \xi^+} - \left.\frac{\partial G}{\partial x}\right|_{x = \xi^-} = -\frac{1}{r(\xi)}$$

where r(x) is the co – efficient of $u''(x)$ in equation (1).

Proof. We have $\left.\frac{\partial G}{\partial x}\right|_{\substack{x = \xi^+ \\ (x > \xi)}} - \left.\frac{\partial G}{\partial x}\right|_{\substack{x = \xi^- \\ (x < \xi)}} = \frac{1}{\beta}\left[v_1'(x)\, v_2(\xi)\right]_{x = \xi} - \frac{1}{\beta}\left[v_2'(x)\, v_1(\xi)\right]_{x = \xi}$

$$= \frac{1}{\beta}\left[v_1'(\xi)\, v_2(\xi) - v_2'(\xi)\, v_1(\xi)\right]$$

$$= \frac{1}{\beta}\left[\frac{-\beta}{r(\xi)}\right] = -\frac{1}{r(\xi)} \qquad \text{[By equation (11)]}$$

3.6. Fredholm Integral Equation and Green's Function. Consider the general boundary value problem

$$A_0(x)\frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_2(x)\, y + \lambda\, p(x)\, y = h(x) \qquad (1)$$

with boundary conditions: y(a) = 0, y(b) = 0. (2)

We shall show that it reduces to Fredholm integral equation with the Green's function as its kernel.

To make the above operator in (1) as a self – adjoint operator, we shift the term λ p(x)y to the right side and then divide it by $\frac{r(x)}{A_0(x)}$.

The solution of (1) in terms of Green's function is

$$y(x) = -\int_a^b G(x, \xi)\; f(\xi)\, d\xi \text{ where } f(x) = h(x) - \lambda\, p(x)\, y(x)$$

or $$y(x) = -\int_a^b G(x, \xi)\; \left[h(\xi) - \lambda\, p(\xi)\, y\right] d\xi$$

$$= -\int_a^b G(x, \xi)\; h(\xi)\, d\xi + \lambda\int_a^b G(x, \xi)\; p(\xi)\, y(\xi)\, d\xi$$

$$= K(x) + \lambda \int_a^b G(x, \xi)\; p(\xi)\, y(\xi)\, d\xi \qquad (3)$$

where $$K(x) = -\int_a^b G(x, \xi)\; h(\xi)\, d\xi \qquad (4)$$

This is a Fredholm integral equation of the second kind with kernel K(x, ξ) = G(x, ξ) p(ξ) and a non – homogeneous term K(x).

Now, multiplying equation (3) by $\sqrt{p(x)}$, we get

$$\sqrt{p(x)}\; y(x) = \sqrt{p(x)}\; K(x) + \lambda\int_a^b \sqrt{p(x)\, p(\xi)}\; G(x, \xi)\sqrt{p(\xi)}\; y(\xi)\, d\xi$$

Let us use, u(x) = $\sqrt{p(x)}$ y(x) and g(x) = $\sqrt{p(x)}$ K(x)

Then, $u(x) = g(x) + \lambda\int_a^b \sqrt{p(x)\,p(\xi)}\; G(x, \xi)\, u(\xi)\, d\xi$ (5)

Here the kernel of Fredholm integral equation of second kind is symmetric that is,

$$K(x, \xi) = \sqrt{p(x)\,p(\xi)}\; G(x, \xi) \quad (6)$$

is symmetric, since $G(x, \xi)$ is symmetric.

Remark : We had obtained the required result in equation (3). We had proceed to obtain equation (5) just to get the kernel in more symmetric form.

3.7. Check Your Progress.

1. Solve the boundary value problem using Green's function $\frac{d^2u}{dx^2} - u = -2e^x$ with boundary conditions $u(0) = u'(0)$, $u(l) + u'(l) = 0$.

Answer. $G(x, \xi) = \begin{cases} \frac{1}{2}e^{x-\xi} & , \; 0 \le x < \xi \\ \frac{1}{2}e^{-(x-\xi)} & , \; \xi < x \le l \end{cases}$ and $u(x) = -[(l - x)e^x + \sinh x]$

2. Solve the boundary value problem using Green's function $\frac{d^4u}{dx^4} = 1$, with boundary conditions $u(0) = u'(0) = u''(1) = u'''(1) = 0$.

Answer. $G(x, \xi) = \begin{cases} \frac{1}{6}x^2(3\xi - x) & , \; 0 \le x < \xi \\ \frac{1}{6}\xi^2(3\xi - x) & , \; \xi < x \le 1 \end{cases}$ and $u(x) = \frac{1}{24}x^2(x^2 - 4x + 6)$.

3.8. Summary. In this chapter, we discussed various methods to construct Green function for a given non-homogeneous linear second order boundary value problem and then boundary value problem can be reduced to Fredholm integral equation with Green function as kernel and hence can be solbed using the methods studied in the previous chapter.

Books Suggested:

1. Jerri, A.J., Introduction to Integral Equations with Applications, A Wiley-Interscience Publication, 1999.
2. Kanwal, R.P., Linear Integral Equations, Theory and Techniques, Academic Press, New York.
3. Lovitt, W.V., Linear Integral Equations, McGraw Hill, New York.
4. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.

4

Calculus of Variations

Structure

4.1. Introduction. This chapter contains methods to obtain extremum of a given functional in one variable, several variable, for functional involving higher derivatives and variational derivatives.

4.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Brachistochrone problem.
(ii) Geodesics Problem.
(iii) Isoperimetric Problem.
(iv) The problem of minimum surface of revolution.

4.1.2. Keywords. Fumctional, extremal, Euler Equation.

4.2. Functional. Let there be a class of functions. By a functional, we mean a correspondence which assigns a definite real number to each function belonging to the class. In other words, a functional is a kind of function where the independent variable is itself a function. Thus, the domain of the functional is the set of functions.

Examples. (1) Let y(x) be an arbitrary continuously differentiable function defined on interval [a, b]. Then the formula,

$J[y] = \int_a^b y'^2(x)\,dx$ defines a functional on the set of all such functions y(x).

(2) Consider the set of all rectifiable plane curves. The length of any curve between the points (x_0, y_0) and (x_1, y_1) on the curve y = y(x) is given by

$$l\left(y(x)\right) = \int_{x_0}^{x_1} \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

This defines a functional on the set of rectifiable curves.

(3) Consider all possible paths joining two given points A and B in the plane. Suppose that a particle can move along any of these paths and let the particle have a definite velocity v (x, y) at the point (x, y). Then we can define a functional by associating with each path, the time taken by the particle to traverse the path.

(4) The expression,

$$J\,[y] = \int_a^b F[x\,,\,y(x)\,,\,y'(x)]dx$$

gives a general example of functional. By choosing different functions $F\,(x,\, y,\, y')$ we obtain different functions. For example, if we take $F \equiv y'^2$, we obtain example (1) given above.

Remark. The generalised study of functionals is called "calculus of functionals". The most developed branch of calculus of functionals is concerned with finding the maxima and minima of functionals and is called "calculus of variation". We shall study this particular branch and shall find the extremals of the functionals.

Further, while finding the extremals we shall consider only the functionals of the type

$$J\,[y] = \int_a^b F[x\,,\,y(x)\,,\,y'(x)]dx$$

where y(x) ranges over the set of all continuously differentiable functions defined on the interval [a, b].

4.2.1. Motivating Problems. The following are some problems involving the determination of maxima and minima of functionals. These problems motivated the development of the subject.

1. **Brachistochrone problem.** Let A and B be two fixed points. Then the time taken by a particle to slide under the influence of gravity along some path joining A and B depends on the choice of the path (curve) and hence is a functional. The curve such that the particle takes the least time to go from A to B is called “brachistochrone”.

The brachistochrone problem was posed by John Bernoulli in 1696 and played an important part in the development of the calculus of variation. The problem was solved by John Bernoulli, James Bernoulli, Newton and L Hospital. the brachistocharone comes out to be a cycloid lying in the vertical plane and passing through A and B.

2. **Geodesics Problem.** In this problem, we have to determine the line of shortest length connecting two given points (x_0, y_0, z_0) and (x_1, y_1, z_1) on a surface S given by ϕ (x, y, z) = 0.

Mathematically, we are required to minimize the arc length *l* joining the two points on S given by

$$l = \int_{x_0}^{x_1} \left[1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right]^{1/2} dx$$

Subject to the constraint $\phi(x, y, z) = 0$

3. Isoperimetric Problem. In this problem, we have to find the extremal of a functional under the constraint that another functional assumes a constant value.

Mathematically, to make

$$J[y] = \int_{x_0}^{x_1} F[x, y(x), y'(x)]dx$$

maximum or minimum such that the functional

$$\phi[y] = \int_{x_0}^{x_1} G[x, y(x), y'(x)]dx \text{ is kept constant.}$$

For example, “Among all closed curves of a given length l, find the curve enclosing the greatest area.” This problem is an isoperimetric problem. This was solved by Euler and required curve comes out to be a circle.

4. **The problem of minimum surface of revolution.** In this problem, we have to find a curve y = y(x) passing through two given points (x_0, y_0) and (x_1, y_1) which when rotated about the x – axis gives a minimum surface area.

Mathematically, the surface area bounded by such curve is given by,

$$S = \int_{x_0}^{x_1} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Thus, we have to find a particular curve y = y(x) which minimizes S.

4.2.2. Function Spaces. As in study of functions, we use geometric language by taking a set of n numbers $(x_1, x_2, \ldots, x_n)$ as a point in an n – dimensional space. In the same way in the study of functionals, we shall regard each functions y(x) belonging to some class as a point in some space and the spaces whose elements are functions are called "Function spaces".

Remark. The concept of continuity plays an important role for functionals, just as it does for the ordinary functions. In order to formulate this concept for functionals, we must somehow introduce a concept of closeness for elements in a function space. This is most conveniently done by introducing the concept of the "norm" of a function (analogous to the concept of distance in Euclidean space). For this, we introduce the following basic concepts starting with a linear space.

4.2.3. Linear space. By a linear space, we mean a set $\mathbb{R}$ of elements x, y, z, … of any kind for which the operations of addition and multiplication by real numbers $\alpha, \beta, \ldots$ are defined and obey the following axioms:

i. $x + y = y + x$

ii. $(x + y) + z = x + (y + z)$

iii. There exists an element '0' such that $x + 0 = x = 0 + x$ for all $x \in \mathbb{R}$

iv. For each $x \in \mathbb{R}$, there exists an element $-x$ s.t. $x + (-x) = 0 = (-x) + x$

v. $1.x = x$

vi. $\alpha(\beta x) = (\alpha\beta)x$

vii. $(\alpha + \beta)x = \alpha x + \beta x$

viii. $\alpha(x + y) = \alpha x + \alpha y$

4.2.4. Normed Linear space. A linear space $\mathbb{R}$ is said to be normed if each element $x \in \mathbb{R}$ is assigned a non negative number $\|x\|$, called the norm of x, such that

(1) $\|x\| = 0$ if and only if $x = 0$

(2) $\|\alpha x\| = |\alpha| \, \|x\|$

(3) $\|x + y\| \le \|x\| + \|y\|$

In a normed linear space, we can talk about distances between elements by defining the distance between x and y to be the quantity $\|x - y\|$.

4.2.5. Important Normed Linear spaces. Here are some examples of normed linear spaces which will be commonly used in our further study.

(1) The space C [a, b] : The space consisting of all continuous functions y(x) defined on a closed interval [a, b] is denoted as C [a, b]. By addition of elements of C [a, b] and multiplication of elements of C[a, b] by numbers, we mean ordinary addition of functions and multiplication of functions by numbers.

The norm in C [a, b] is defined as:

$$\|y\|_0 = \max_{a \le x \le b} . |y(x)|$$

(2) The space D [a, b] : This space consists of all functions y(x) defined on an interval [a, b] which are continuous and have continuous first derivatives. The operations of addition and multiplication by numbers are the same as in C [a, b] but the norm is defined by the formula,

$$\|y\|_1 = \max.\ a \le x \le b\ |y(x)| + \max.\ a \le x \le b\ |y'(x)|$$

Thus, two functions in D [a, b] are regarded as close together if both the functions themselves and their first derivatives are close together, since

$$\|y - z\|_1 < \in \text{ implies that } |y(x) - z(x)| < \in \text{ and } |y'(x) - z'(x)| < \in .$$

(3) The Space D_n [a, b] : The space D_n [a, b] consists of all functions y(x) defined on an interval [a, b] which are continuous and have continuous derivatives upto order n where n is fixed integer. Addition of elements of D_n and multiplication of elements of D_n by numbers are defined just as in the preceding cases, but the norm is defined as :

$$\|y\|_n = \sum_{i=0}^{n} \max.\ a \le x \le b \left| y^{(i)}(x) \right|$$

where $y^{(i)}(x) = \left(\frac{d}{dx}\right)^i$ y(x) and $y^{(0)}$ (x) denotes the function y(x) itself.

Thus two functions in D_n [a, b] are regarded as close together if the values of the functions themselves and of all their derivatives upto order n inclusive are close together.

4.2.6. Closeness of functions.

(1) The functions y(x) and z(x) are said to be close in the sense of zero order proximity if the value $|y(x) - z(x)|$ is small for all x for which the functions are defined. Thus, in the space C [a, b], the closeness is in the sense of zero order proximity.

(2) The functions y(x) and z(x) are said to be close in the sense of first order proximity if both $|y(x) - z(x)|$ and $|y'(x) - z'(x)|$ are small for all values of x for which the functions are defined. Thus, in the space D [a, b], the closeness is in the sense of first order proximity.

(3) The functions y(x) and z(x) are said to be close in the sense of n^{th} order proximity if

$|y(x) - y(x)|$, $|y'(x) - z'(x)|, \ldots,$ $|y^n(x) - z^n(x)|$ are small for all values of x for which the functions are defined. In the space D_n [a, b], the closeness is in the sense of n^{th} order proximity.

4.2.7. Continuity of functional.

The functional J [y(x)] is said to be continuous at the point $y = y_0(x)$ if for any $\in > 0$, there exists a $\delta > 0$ such that

$$|J[y] - J[y_0]| < \in \text{ whenever } \|y - y_0\| < \delta$$

Further, the continuity is said to be in the sense of zero, first or n^{th} order proximity according as norm is defined as in C [a, b], D [a, b] or D_n [a, b].

4.2.8. Linear functional : Given a normed linear space $\mathbb{R}$, let each element $h \in \mathbb{R}$ be assigned a number ϕ [h] that is, let ϕ [h] be a functional defined on $\mathbb{R}$. Then ϕ [h] is said to be a linear functional if

(i) $\phi[\alpha h] = \alpha \phi[h]$ for any $h \in \mathbb{R}$ and any real number α.

(ii) $\phi[h_1 + h_2] = \phi[h_1] + \phi[h_2]$ for any $h_1, h_2 \in \mathbb{R}$.

For example,

(1) If we associate with each function $h(x) \in C[a, b]$ its value at a fixed point x_0 in [a, b] that is, if we define the functional $\phi[h]$ by the formula $\phi[h] = h(x_0)$ then $\phi[h]$ is a linear functional on C [a, b].

(2) The integral $\phi[h] = \int_a^b h(x)\, dx$ defines a linear functional on C [a, b].

(3) The integral $\phi[h] = \int_a^b \alpha(x)\, h(x)\, dx$ where $\alpha(x)$ is a fixed function in C [a, b] defines a linear functional on C [a, b].

(4) More generally, the integral

$$\phi[h] = \int_a^b \left[\alpha_0(x)\, h(x) + \alpha_1(x)\, h^1(x) + \ldots + \alpha_n(x)\, h^n(x)\right] dx$$

where the $\alpha_i(x)$ are fixed functions in C [a, b] defines a linear functional on D_n [a, b]

4.2.9. Lemma. If $\alpha(x)$ is continuous in [a, b] and if $\int_a^b \alpha(x)\, h(x)\, dx = 0$ for every function $h(x) \in C[a, b]$ such that $h(a) = h(b) = 0$, then $\alpha(x) = 0$ for all $x \in [a, b]$

Proof. Let, if possible, the function $\alpha(x)$ be non – zero say positive, at some point of [a, b]. Then by virtue of continuity, $\alpha(x)$ is also positive in some interval $[x_1, x_2] \subseteq [a, b]$.

We set,

$$h(x) = \begin{cases} (x - x_1)(x_2 - x), & \text{for all } x \in [x_1, x_2] \\ 0, & \text{otherwise} \end{cases}$$

Then, h(x) obviously satisfies the conditions of the lemma. But we have

$$\int_a^b \alpha(x)\, h(x)\, dx = \int_{x_1}^{x_2} \alpha(x)(x - x_1)(x_2 - x)\, dx > 0,$$

since the integrand is positive (except at x_1 and x_2). This contradiction proves the lemma.

Remark. The lemma still holds if we replace C [a, b] by D_n [a, b].

4.2.10. Lemma. If α (x) is continuous in [a, b] and if $\int_a^b \alpha(x)\, h'(x)\, dx = 0$, for every h(x) in D_1[a, b] such that h(a) = h(b) = 0, then α (x) = C for all x $\in$ [a, b] where C is a constant.

Proof. Let C be the constant defined by the condition

$$\int_a^b [\alpha(x) - C]\, dx = 0 \qquad (1)$$

which, in fact, gives C = $\dfrac{1}{b - a} \displaystyle\int_a^b \alpha(x)\, dx$

Also, let h(x) = $\int_a^b [\alpha(\xi) - C]\, d\xi$ then clearly h(x) $\in$ D_1 [a, b]. Also we have,

$$h(a) = \int_a^b [\alpha(\xi) - C]\, d\xi = 0 \text{ and } h(b) = \int_a^b [\alpha(\xi) - C]\, d\xi = 0 \qquad \text{[By (1)]}$$

Thus h(x) satisfies all the conditions of the lemma and so by given hypothesis.

$$\int_a^b \alpha(x)\, h'(x)\, dx = 0 \qquad (2)$$

Now we calculate

$$\int_a^b [\alpha(x) - C]\, h'(x)\, dx = \int_a^b \alpha(x)\, h'(x)\, dx - C\int_a^b h'(x)\, dx = 0 - C\int_a^b h'(x)\, dx \qquad \text{[By (2)]}$$

$$= -C\,[h(b) - h(a)] = -C\,[0 - 0] = 0 \qquad (3)$$

On the other hand, by definition of h(x)

$$\int_a^b [\alpha(x) - C]\, h^1(x)\, dx = \int_a^b [\alpha(x) - C][\alpha(x) - C]\, dx = \int_a^b [\alpha(x) - C]^2\, dx \qquad (4)$$

Expression (3) and (4) give the value of same integral so we have

$$\int_a^b [\alpha(x) - C]^2\, dx = 0 \Rightarrow \alpha(x) - C = 0 \qquad \Rightarrow \alpha(x) = C \text{ for all } x \in [a, b]$$

4.2.11. Lemma. If $\alpha(x)$ is continuous in [a, b] and if $\int_a^b \alpha(x)\, h''(x)\, dx = 0$, for every function $h(x) \in D_2$ [a, b] such that h(a) = h(b) = 0 and $h'(a) = h'(b) = 0$. Then $\alpha(x) = C_0 + C_1 x$ for all $x \in [a, b]$ where C_0 and C_1 are constants.

Proof. Let C_0 and C_1 be defined by the conditions

$$\int_a^b [\alpha(x) - C_0 - C_1 x]\, dx = 0 \qquad (1)$$

$$\int_a^b \int_a^x [\alpha(\xi) - C_0 - C_1 \xi]\, d\xi\, dx = 0 \qquad (2)$$

and let $h(x) = \int_a^x \int_a^\xi [\alpha(t) - C_0 - C_1 t]\, dt\, d\xi \qquad (3)$

$$\Rightarrow \qquad h'(x) = \int_a^x [\alpha(t) - C_0 - C_1 t]\, dt \qquad (4)$$

and $\qquad h''(x) = \alpha(x) - C_0 - C_1 x \qquad (5)$

Then, clearly $h(x) \in D_2[a, b]$.

Also, we have

$$h(a) = \int_a^a \int_a^\xi [\alpha(t) - C_0 - C_1 t]\, dt\, d\xi = 0$$

$$h(b) = \int_a^b \int_a^\xi [\alpha(t) - C_0 - C_1 t]\, dt\, d\xi = 0 \qquad \text{[By (2)]}$$

$$h'(a) = \int_a^a [\alpha(t) - C_0 - C_1 t]\, dt = 0 \text{ and } h'(b) = \int_a^b [\alpha(t) - C_0 - C_1 t]\, dt = 0 \text{ [By (1)]}$$

Thus h(x) satisfies all the conditions of the lemma and so by given hypothesis,

$$\int_a^b \alpha(x)\, h''(x)\, dx = 0 \qquad (6)$$

Now we calculate,

$$\int_a^b [\alpha(x) - C_0 - C_1 x]\, h''(x)\, dx = \int_a^b \alpha(x)\, h''(x)\, dx - C_0 \int_a^b h''(x)\, dx - C_1 \int_a^b x\, h''(x)\, dx$$

$$= 0 - C_0\, [h'(b) - h'(a)] - C_1 \left[x\big(h'(b) - h'(a)\big) - \int_a^b h'(x)\, dx \right]$$

$$= 0 - C_0(0) - C_1\big[(0) - \big(h(b) - h(a)\big)\big] = 0 \qquad (7)$$

On the other hand,

$$\int_a^b [\alpha(x) - C_0 - C_1 x]\, h''(x)\, dx = \int_a^b [\alpha(x) - C_0 - C_1 x]^2\, dx \ \text{[By (5)]} \qquad (8)$$

By (7) and (8), it follows that

$\alpha(x) - C_0 - C_1x = 0 \Rightarrow \alpha(x) = C_0 + C_1x$ for all x in [a, b].

4.2.12. Lemma. If $\alpha(x)$ and $\beta(x)$ are continuous in [a, b] and if

$$\int_a^b [\alpha(x)\, h(x) + \beta(x)\, h'(x)]\, dx = 0$$

for every function $h(x) \in D_1[a, b]$ such that h(a) = h(b) = 0 then prove that $\beta(x)$ is differentiable and $\beta'(x) = \alpha(x)$ for all x in [a, b].

Proof. Let us set $A(x) = \int_a^x \alpha(\xi)\, d\xi$. Now integrating by parts, the integral $\int_a^b \alpha(x)\, h(x)\, dx$, we get

$$\int_a^b \alpha(x)\, h(x)\, dx = \left[h(x) \int_a^x \alpha(x)\, dx \right]_a^b - \int_a^b h'(x) \int_a^x \alpha(x)\, dx$$

$$= [h(x)\, A(x)]_a^b - \int_a^b h'(x)\, A(x)\, dx$$

$$= h(b)\, A(b) - h(a)\, A(a) - \int_a^b h'(x)\, A(x)\, dx = 0 - 0 - \int_a^b h'(x)\, A(x)\, dx$$

$$\Rightarrow \quad \int_a^b \alpha(x)\, h(x)\, dx = -\int_a^b A(x)\, h'(x)\, dx \qquad (1)$$

Now it is given that,

$$\int_a^b \left[\alpha(x)\, h(x) + \beta(x)\, h'(x)\right] dx = 0$$

Using (1), it becomes

$$\int_a^b \left[-A(x)\, h'(x) + \beta(x)\, h'(x)\right] dx = 0 \Rightarrow \int_a^b \left[\beta(x) - A(x)\right] h'(x) = 0 \text{ for all } x \in [a, b]$$

Thus using lemma (2), we get,

$$\beta(x) - A(x) = C, \text{ (a constant).}$$

$$\Rightarrow \quad \beta(x) = A(x) + C \Rightarrow \beta(x) = \int_a^x \alpha(\xi)\, d\xi + C \Rightarrow \beta'(x) = \alpha(x) \text{ for all } x \in [a, b]$$

Hence proved.

Remark. The basic work is over. We now introduce the concept of the variation (or differential) of a functional. The concept will the be used to find extrema of functionals.

4.3. Variation of a functional. Let J [y] be a functional defined on some normed linear space and let $\Delta J[h] = J[y + h]$ be its increment corresponding to the increment $h = h(x)$ of the independent variable $y = y(x)$. If y is fixed $\Delta J[h]$ is a functional of h.

Suppose that $\Delta J[h] = \phi[h] + \in \|h\|$ where $\phi[h]$ is a linear functional and $\in \to 0$ as $\|h\| \to 0$. Then the functional J [y] is said to be differentiable and the principal linear part of the increment $\Delta J[h]$ i.e the linear functional $\phi[h]$ is called variation (or differential) of J[y] and is denoted by $\delta J[h]$. Thus, we can write

$$\Delta J[h] = \delta J[h] + \in \|h\| \text{ where } \in \to 0 \text{ as } \|h\| \to 0.$$

4.3.1. Theorem. The differential of a differentiable functional is unique.

Proof. Let $\phi[h]$ be a linear functional such that $\dfrac{\phi[h]}{\|h\|} \to 0$ as $\|h\| \to 0$, then we claim that $\phi[h] \equiv 0$ for all h.

Let, if possible, $\phi[h_0] \neq 0$ for some $h_0 \neq 0$, then by setting $h_n = \dfrac{h_0}{n}$ and $\lambda = \dfrac{\phi[h_0]}{\|h_0\|}$ we observe that

$\|h_n\| \to 0$ as $n \to \infty$ but $\displaystyle\lim_{n\to\infty} \frac{\phi[h_n]}{\|h_n\|} = \lim_{n\to\infty} \frac{n\,\phi[h_0]}{n\,\|h_0\|} = \lambda \neq 0$ which is a contradiction to the hypothesis.

Now, we prove the uniqueness of differential. Let J [y] be a differentiable functional and let, if possible, ϕ_1[h] and ϕ_2[h] be two variations of J [y], then

$$\Delta\ J[y] = \phi_1[h] + \in_1 \|h\|; \qquad \in_1 \to 0 \text{ as } \|h\| \to 0$$

$$\Delta\ J[y] = \phi_2[h] + \in_2 \|h\|; \qquad \in_2 \to 0 \text{ as } \|h\| \to 0$$

where both ϕ_1[h] and ϕ_2[h] are linear functionals.

Subtracting these, we get

$$\phi_1[h] - \phi_2[h] = \in_2 \|h\| - \in_1 \|h\| \qquad \Rightarrow \frac{\phi_1[h] - \phi_2[h]}{\|h\|} = \in_2 - \in_1$$

Taking limit $\|h\| \to 0$, we get,

$$\lim_{\|h\|\to 0} \frac{\phi_1[h] - \phi_2[h]}{\|h\|} = 0 \qquad\qquad [\text{Since } \in_1, \in_2 \to 0 \text{ as } \|h\| \to 0]$$

By above part, we get $\phi_1[h] - \phi_2[h] \equiv 0 \Rightarrow \phi_1[h] = \phi_2[h]$

Hence the uniqueness.

Remark. Let us recall the concept of extremum from analysis.

Let F ($x_1, x_2, \ldots, x_n$) be a differentiable function of n variables. then F($x_1, x_2, \ldots, x_n$)

is said to have an extremum at the point $(x_1', x_2', ..., x_n')$ if

$$\Delta F = F(x_1, x_2, \ldots, x_n) - F(x_1', x_2', ..., x_n')$$

has the same sign for all points ($x_1, x_2, \ldots, x_n$) belonging to some neighbourhood of $(x_1', x_2', ..., x_n')$. Further, the extremum is a minimum if $\Delta F \geq 0$ and is a maximum if $\Delta F \leq 0$.

4.3.2. Extremum of a functional. We say that the functional J [y] has an extremum for y = y if J [y] – J [y] does not change sign in some neighbourhood of the curve y = y (x).

Depending upon whether the functional are the elements of C [a, b] or D[a, b], we define two kinds of extrema:

1. Weak Extremum. We say that the functional J [y] has a weak extremum for y = y if there exists an $\in > 0$ such that J [y] – J [y] has the same sign for all y in the domain of the definition of the functional which satisfy the condition $\|y - y\|_1 < \in$ where $\| \ \|_1$ denotes the norm in D_1 [a, b].

2. Strong Extremum. We say that the functional J [y] has a strong extremum for y = y if there exists an $\in$ > 0 such that J [y] – J [y] has the same sign for all y in the domain of definition of the functional which satisfy the condition $\|y - y\|_0 < \in$ where $\| \ \|_0$ denotes the norm in the space C [a, b].

Remark. It is clear by definitions that every strong extremum is simultaneously a weak extremum since if $\|y - y\|_1 < \in$, then $\|y - y\|_0 < \in$ and hence if J [y] is an extremum w.r.t all y such that $\|y - y\|_0 < \in$, then J [y] is certainly an extremum w.r.t. all y such that $\|y - y\|_1 < \in$. However, the converse is not true, in general.

4.3.3. Admissible functions. The set of functions satisfying the constraints of a given variational problem are called admissible functions of that variational problem.

4.3.4. Theorem. A necessary condition for the differentiable functional J [y] to have an extremum for y = y, is that its variation vanish for y = y that is, that δ J [h] = 0 for y = y and all admissible h.

Proof. W.L.O.G., suppose J [y] has a minimum for y = y so that

$$\Delta\ J[h] = J[y + h] - J[y] \geq 0 \text{ for all sufficiently small } \|h\|. \qquad (1)$$

Now by definition we have,

$$\Delta\ J[h] = \delta\ J[h] + \in \|h\| \text{ where } \in \to 0 \text{ as } \|h\| \to 0. \qquad (2)$$

Thus for sufficiently small $\|h\|$, the sign of Δ J [h] will be the same as the sign of δ J [h].

Now, suppose that if possible δ J [h_0] $\neq$ 0 for some admissible h_0.

Then for any α > 0, no matter however small,

$$\delta J[-\alpha\ h_0] = -\delta J[\alpha h_0]$$

Thus by (2), ΔJ[h] can be made to have either sign for arbitrary small $\|h\|$ which is a contradiction to (1). Hence δ J [h] = 0 for y = y and all admissible h.

4.3.5. Euler's Equation. Let J [y] be a functional of the form $\int_a^b F(x, y, y')\,dx$ defined on the set of functions y(x) which have continuous first derivatives in [a, b] and satisfy the boundary condition y (a) = A, y(b) = B. Then a necessary condition for J [y] to have an extremum for a given function y(x) is that y(x) satisfy the equation

$$F_y - \frac{d}{dx}(F_{y'}) = 0$$

Proof : Suppose we give y(x) an increment h(x) where in order for the function y(x) + h(x) to continue to satisfy the boundary conditions, we must have h(a) = h(b) = 0.

We calculate corresponding increment to the given functional.

$$\Delta\ J = J\,[y + h] - J\,[y] = \int_a^b F(x\,,\,y + h\,,\,y' + h')\,dx \;-\; \int_a^b F(x\,,\,y\,,\,y')\,dx$$

$$= \int_a^b \left[F\left(x\,,\,y + h\,,\,y' + h'\right) - F\,(x\,,\,y\,,\,y')\right] dx$$

Using Taylor’s theorem, we obtain,

$$\Delta\ J = \int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right]dx \;+\; \begin{bmatrix}\text{terms containing higher}\\ \text{order partial derivatives}\\ \text{and powers of h and h}'\\ \text{greater than 1}\end{bmatrix}$$

We express this as

$$\Delta\ J = \int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right]dx \;+\; \ldots$$

Clearly the integral $\int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right]dx$ represents the principal linear part of $\Delta\ J$ and hence, we write,

$$\delta\ J = \int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right]dx$$

Now by theorem (2), the necessary condition for J [y] to be extremum is that δ J = 0, so that

$$\int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right]dx \;=\; 0$$

By lemma (4) (proved earlier), we obtain that

$$F_y = \frac{d}{dx}\,(F_{y'}) \qquad\qquad [\text{Take } F_y = \alpha\,(x) \text{ and } F_{y'} = \beta\,(x) \text{ in lemma (4)}]$$

$$\Rightarrow \qquad F_y - \frac{d}{dx}\,(F_{y'}) = 0.$$

This equation is known as Euler’s Equation.

4.3.6. Another form of Euler's equation : As F is a function of x, y and y', so we have :

$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' \qquad (1)$$

Also we have,

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}y'' \qquad (2)$$

Subtracting (2) from (1), we obtain,

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y}\right)$$

which can be written as,

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial x} = y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] = y'\,[0] \text{ [By Euler's equation]}$$

$$\Rightarrow \quad \frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial x} = 0 \text{ or } \frac{d}{dx}\left(F - y'F_{y'}\right) - F_x = 0$$

This is another form of Euler equation.

Remark. Euler's equation

$$F_y - \frac{d}{dx}(F_{y'}) = 0 \qquad (*)$$

and

$$\frac{d}{dx}\left(F - y'F_{y'}\right) - F_x = 0 \qquad (**)$$

plays a fundamental role in the calculus of variations and is in general a second order differential equation we now discuss some special cases:

Case 1. Suppose the integrand does not depend on y that is, let the functional of under consideration has the form $\int_a^b F(x, y')\,dx$, so that we have $F_y = 0$

Then by (*), $\frac{d}{dx}(F_{y'}) = 0 \Rightarrow F_{y'} = C$, a constant which is a first order differential equation and can be solved by integration.

Case 2. If the integrand does not depend on x that is, the functional has the form

$\int_a^b F(y, y')\,dx$ then $F_x = 0$ and so by (**), $\frac{d}{dx}\left(F - y'F_{y'}\right) = 0 \quad \Rightarrow F - y'F_{y'} = C$

Case 3. If F does not depend upon y', then again by (*), we get, $F_y = 0$ which is not a differential equation but a finite equation whose solution consists of one or more curves $y = y(x)$.

Case 4. Consider the functional of the form

$$\int_a^b f(x, y)\, ds = \int_a^b f(x, y)\sqrt{1+y'^2}\, dx$$

representing the integral of a function f (x, y) w.r.t. the arc length s where $ds = \sqrt{1+y'^2}\,dx$. In this case we have $F(x, y, y') = f(x, y)\sqrt{1+y'^2}$ and Euler's equation becomes,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = f_y(x, y)\sqrt{1+y'^2} - \frac{d}{dx}\left[f(x, y)\frac{y'}{\sqrt{1+y'^2}}\right]$$

$$= f_y\sqrt{1+y'^2} - f_x\frac{y'}{\sqrt{1+y'^2}} - f_y\frac{y'^2}{\sqrt{1+y'^2}} - f\frac{y''}{\left(1+y'^2\right)^{3/2}} = \frac{f_y}{\sqrt{1+y'^2}} - \frac{f_x y'}{\sqrt{1+y'^2}} - \frac{f y''}{\left(1+y'^2\right)^{3/2}}$$

$= 0$

Thus, $f_y - f_x\, y' - f\,\dfrac{y''}{1+y'^2} = 0$ which is the required form of Euler's equation.

4.3.7. Example. Find the extremal of the functional $J[y] = \int_1^2 \frac{\sqrt{1+y'^2}}{x}\,dx$ $y(1) = 0$, $y(2) = 1$.

Solution. Since the integrand does not contain y, so we shall use Euler's equation in the form

$$F_{y'} = \text{constant} = c \text{ (say)} \qquad (*)$$

Now, we have

$$F = \frac{\sqrt{1+y'^2}}{x} \Rightarrow F_{y'} = \frac{1}{2}\cdot\frac{2y'}{\sqrt{1+y'^2}}\cdot\frac{1}{x} = \frac{y'}{x\sqrt{1+y'^2}}$$

Using this in (*), we get

$$\frac{y'}{x\sqrt{1+y'^2}} = c \Rightarrow \left(\frac{dy}{dx}\right)^2 = c^2x^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow (1 - C^2x^2)\left(\frac{dy}{dx}\right)^2 = c^2x^2 \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{c^2x^2}{1-c^2x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{cx}{\sqrt{1-c^2x^2}}$$

$$\Rightarrow \quad y = \int \frac{cx}{\sqrt{1-c^2x^2}} + c' = \int \frac{x}{\sqrt{\frac{1}{c^2}-x^2}}\,dx + c' = -\frac{1}{2}\int \frac{-2x}{\sqrt{\frac{1}{c^2}-x^2}}\,dx + c'$$

$$= -\frac{1}{2}\int \frac{\left(\frac{1}{c^2}-x^2\right)^{1/2}}{1/2} + c' = -\sqrt{\frac{1}{c^2}-x^2} + c'$$

$$\Rightarrow \quad (y - c')^2 = \frac{1}{c^2}-x^2 \Rightarrow (y - c')^2 + x^2 = k^2, \text{ say}$$

Thus, the solution is a circle with its centre on the y – axis. Using the conditions

$$y(1) = 0,\ y(2) = 1,$$

we find that $c' = 2$, $k = \sqrt{5}$

So that the final solution is, $(y-2)^2 + x^2 = 5$.

4.3.8. Example. Among all the curves joining two given points (x_0, y_0) and (x_1, y_1). Find the one which generates the surface of minimum area when rotaed about the x – axis.

Solution. We know (from calculus) that the area of surface of revolution generated by rotating the curve $y = y(x)$ about the x – axis is given by :

$$S = \int_{x_0}^{x_1} 2\pi\, y\sqrt{1+y'^2}\,dx = 2\pi \int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx$$

Since the integrand does not depend explicitly on x, Euler's equation can be written as :

$$F - y' F_{y'} = C, \text{ constant.}$$

$$\Rightarrow \quad y\sqrt{1+y'^2} - y\frac{y'^2}{\sqrt{1+y'^2}} = C \qquad \Rightarrow y\left[\frac{1+y'^2-y'^2}{\sqrt{1+y'^2}}\right] = C \Rightarrow \frac{y^2}{1+y'^2} = C^2$$

$$\Rightarrow \quad y' = \frac{\sqrt{y^2-C^2}}{C} \Rightarrow \frac{dy}{dx} = \frac{1}{C}\sqrt{y^2-C^2} \Rightarrow dx = \frac{C\,dy}{\sqrt{y^2-C^2}}$$

$$\Rightarrow \quad x + C_1 = C\cosh^{-1}\left(\frac{y}{C}\right) \Rightarrow y = C\cosh\left(\frac{x+C_1}{C}\right)$$

which is the equation of a catenary. The values of arbitrary constants can be determined by the conclitions

$$y(x_0) = y_0,\ y(x_1) = y_1$$

4.3.9. Example. Find the extremal of the functional $J[y] = \int_a^b (x-y)^2 dx$

Solution. Here $F = (x - y)^2$ which does not contain y' explicitely so that Euler's equation is $F_y = 0$ which gives

$$2(x - y) = 0 \qquad \Rightarrow y = x$$

which is a finite equation and represents a straight line.

4.3.10. Example. Show that the shortest distance between two points in a plane is a straight line.

Solution. Let A (x_1, y_1) and B (x_2, y_2) be the given points and let 's' be the length of curve connecting them, then $\quad S = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$

Here $F = \sqrt{1+y'^2}$ which is independent of y. So Euler's equation is

$$F_{y'} = \text{constant} \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = c \Rightarrow \frac{y'^2}{c^2} = 1+y'^2 \Rightarrow y'^2 = \frac{c^2}{1-c^2} \Rightarrow y' = \frac{c}{\sqrt{1-c^2}} = m \text{ (say)}$$

$$\Rightarrow \quad \frac{dy}{dx} = m \qquad \Rightarrow \qquad y = mx + c, \text{ which is the equation of a straight line.}$$

4.3.11. Exercise.

1. Show that the functional $\int_0^1 (xy + y^2 - 2y^2y')\, dx$, $y(0) = 1$, $y(1) = 2$ cannot have any stationary function.
2. Find extremals of the functional $J[y(x)] = \int_0^{2\pi} (y'^2 - y^2)\, dx$ that satisfy the boundary condition $y(0) = 1$, $y(2\pi) = 1$.

Answer. $y = 1.\cos x + C_2 \sin x$ that is, $y = \cos x + C_2 \sin x$.

3. Obtain the general solution of the Euler's equation for the functional $\int_a^b \frac{1}{y}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

Answer. $(x-h)^2 + y^2 = k^2$.

4. Find the curve on which functional $\int_0^1 \left[(y')^2 + 12xy\right] dx$ with boundary conditions $y(0) = 0$ and $y(1) = 1$ can be extremized.

Answer. $y = x^3$.

4.4. Functionals Dependent on Higher Order Derivatives.

4.4.1. Theorem. A necessary condition for the extremum of a functional of the form $J[y]=\int_a^b F\left[x, y, y', ..., y^{(n)}\right] dx$, where F is differentiable w.r.t. each of its arguments is

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0$$

Proof. Let us consider the functional,

$$J[y] = \int_a^b F\left[x, y, y', ..., y^{(n)}\right] dx$$ satisfying the boundary conditions,

$$y(a) = y_1 ;\ y'(a) = y_1' ;\ \ldots ;\ y^{(n-1)}(a) = y_1^{n-1}$$

$$y(b) = y_2 ;\ y'(b) = y_2' ;\ \ldots ;\ y^{(n-1)}(b) = y_2^{n-1}$$

We give y(x) an increment h(x) so that y(x) + h(x) also satisfies the above boundary conditions. For this, we must have

$$h(a) = h'(a) = \ldots = h^{(n-1)}(a) = 0$$

and $$h(b) = h'(b) = \ldots = h^{(n-1)}(b) = 0 \qquad (1)$$

We now calculate the corresponding increment to the given functional,

$$\Delta J = J[y+h] - J[y]$$

which gives,

$$\Delta J = \int_a^b \left[F\left(x, y+h, y'+h', ..., y^{(n)}+h^{(n)}\right) - F\left(x, y, y', ..., y^{(n)}\right)\right] dx$$

Using Taylor's theorem, we obtain,

$$\Delta J = \int_a^b \left(F_y h + F_{y'} h' + ... + F_{y^{(n)}} h^{(n)}\right) dx + ...$$

The integral on R.H.S. represents the principal linear part of the increment ΔJ and hence the variation of J [y] is

$$\delta J = \int_a^b \left(F_y h + F_{y'} h' + ... + F_{y^{(n)}} h^{(n)}\right) dx$$

Therefore, the necessary condition $\delta J = 0$ for an extemum implies that

$$\int_a^b \left(F_y h + F_{y'} h' + ... + F_{y^{(n)}} h^{(n)}\right) dx = 0 \qquad (2)$$

On R.H.S. of (2), integrate the 2nd term by parts once so that,

$$\int_a^b F_{y'}\, h'\, dx = \left[F_{y'}\, h(x)\right]_a^b - \int_a^b \frac{d}{dx}\left(F_{y'}\right) h(x)\, dx \qquad (3)$$

On R.H.S. of (2), integrated the 3rd term by parts twice to get

$$\int_a^b F_{y''} h''dx = \left[F_{y''}\, h'(x)\right]_a^b - \left[\frac{d}{dx}\left(F_{y''}\right) h(x)\right]_a^b + \int_a^b \frac{d^2}{dx^2}\left(F_{y''}\right) h\, dx \quad (4)$$

Continuing like this, integrating the last term by parts n times, we get

$$\int_a^b F_{y^{(n)}} h^{(n)}\, dx = \left[F_{y^{(n)}}\, h^{(n-1)}(x)\right]_a^b - \left[\frac{d}{dx}\left(F_{y^{(n)}}\right) h^{(n-2)}(x)\right]_a^b + \ldots + (-1)^n \int_a^b \frac{d^n}{dx^n}\left(F_{y^{(n)}}\right) h(x)\, dx \qquad (5)$$

Using the boundary conditions (1) in (3), (4), (5), the integrated parts within the limits a and b vanish and then using these in (2), we get

$$\int_a^b \left[F_y - \frac{d}{dx}\left(F_{y'}\right) + \frac{d^2}{dx^2}\left(F_{y''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}}\right] h(x)\, dx = 0$$

Thus by a lemma proved earlier, we have

$$F_y - \frac{d}{dx}\left(F_{y'}\right) + \frac{d^2}{dx^2}\left(F_{y''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n}\left(F_y\,(n)\right) = 0$$

This result is again called Euler's equation which is a differential equation of order 2n. Its general solution contains 2n arbitrary constants which can be determined by the boundary conditions.

4.4.2. Example. Find the stationary function of the functional $\int_a^b (y'^2 + y\, y'')\, dx$; $y(a) = \lambda_1$ $y'(a) = \lambda_2$, $y(b) = \lambda_3$ $y'(b) = \lambda_4$.

Solution. The given functional is $\int_a^b (y'^2 + y\, y'')\, dx$. Let $F(x, y, y', y'') = y'^2 + y\, y''$.

The Euler's - Poisson Equation is $F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0$ (1)

Here $F_y = y''$ $F_{y'} = 2y'$ $F_{y''} = y$.

So (1) $\Rightarrow$ $y'' - \frac{d}{dx}(2y') + \frac{d^2}{dx^2}(y) = 0$

$y'' - 2y'' + y'' = 0$ $\Rightarrow$ 0 = 0, which is not a differential equation.

So, there is no extremal and hence no stationary function.

4.4.3. Exercise.

1. Find the curve the extremises the functional $\int_0^{\pi/4} (y''^2 - y^2 + x^2)\, dx$ under the boundary condition y(0) = 0, $y'(0) = 1$, $y(\pi/4) = y'(\pi/4) = \frac{1}{\sqrt{2}}$.

Answer. y = sinx

4.5. Functionals dependent on Functions of Several Independent Variables.

So far, we have considered functionals depending on functions of one variable that is, on curves. In many problems, one encounters functionals depending on functions of several independent variables that is, on surfaces. We now, try to find the extremum of such functionals.

However, for simplicity, we confine ourselves to the case of two independent variables. Thus, let F(x, y, z, z_x, z_y) be a function with continuous first and second partial derivatives w.r.t all its arguments and consider a functional of the form

$$J(z) = \iint_R F(x, y, z, z_x, z_y)\,dx\,dy$$

where R is some closed region. Before giving the Euler's equation for such functionals, we prove the following lemma.

4.5.1. Lemma. If α(x, y) is a fixed function which is continuous in a closed region R and if the integral

$$\iint_R \alpha(x, y)\, h(x, y)\,dx\,dy$$

vanishes for every function h (x, y) which has continuous first and second derivatives in R and equals zero on the boundary D of R, then prove that α(x, y) = 0 everywhere in R.

Proof. Let, if possible, the function α(x, y) is non zero, say positive at some point say (x_0, y_0) in R. Then by continuity α(x, y) is also positive in some circle

$$(x - x_0)^2 + (y - y_0)^2 \le \in^2 \qquad (1)$$

contained in R with centre (x_0, y_0) and radius $\in$.

Now we set h(x, y) = $\left[(x - x_0)^2 + (y - y_0)^2 - \in^2\right]^3$ inside the circle (1) and h (x, y) = 0 outside the circle. It is clear that h (x, y) has continuous first and second order derivatives in circle and also h(x) equals zero on the boundary of the circle so that all the conditions of the lemma are satisfied. Hence we must have

$$\iint_{R'} \alpha(x, y)\, h(x, y)\,dx\,dy = 0 \qquad (2)$$

where R' is circle given by (1)

But it is clear that integrand in (2) is positive in circle (1) and so integral (2) is obviously positive.

This contradiction proves the lemma.

Green's Theorem. It states that

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_D Pdx + Qdy\,,$$

where D is boundary of the surface R.

4.5.2. Theorem. A necessary condition for the functional

$$J(z) = \iint_R F(x, y, z, z_x, z_y)dx\, dy$$

to have an extremum for a given function z (x, y) is that z (x, y) satisfies the equation

$$F_z - \frac{\partial}{\partial x} F_{zx} - \frac{\partial}{\partial y} F_{zy} = 0$$

Proof. Let h (x, y) be an arbitrary function which has continuous first and second derivatives in the region R and vanishes on the boundary D of R. Then if z (x, y) belongs to the domain of definition of the given functional so does z (x, y) + h (x, y)

Thus we have,

$$\Delta J = J[z+h] - J[z] = \iint_R F(x, y, z+h, z_x + h_x, z_y + h_y) - F(x, y, z, z_x, z_y)\, dxdy$$

Using Taylor's theorem, we get,

$$\Delta J = \iint_R (F_z h + F_{z_x} h_x + F_{z_y} h_y)\, dxdy + \ldots$$

Integral on R.H.S. represents the principal linear part of the increment ΔJ and hence the variation of J[z] is,

$$\delta J = \iint_R (F_z h + F_{z_x} h_x + F_{z_y} h_y)\, dxdy \qquad (1)$$

Now consider,

$$\frac{\partial}{\partial x}\left(F_{z_x} h\right) = F_{z_x} h_x + \frac{\partial}{\partial x}\left(F_{z_x}\right).h \text{ and } \frac{\partial}{\partial y}\left(F_{z_y} h\right) = F_{z_y} h_y + \frac{\partial}{\partial y}\left(F_{z_y}\right).h$$

which give, $F_{z_x} h_x = \frac{\partial}{\partial x}\left(F_{z_x} h\right) - \frac{\partial}{\partial x}\left(F_{z_x}\right).h$ and $F_{z_y} h_y = \frac{\partial}{\partial y}\left(F_{z_y} h\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right).h$

Using these in (1), we obtain

$$\delta J = \iint_R F_z h\, dxdy + \iint_R \left[\frac{\partial}{\partial x}\left(F_{z_x} h\right) + \frac{\partial}{\partial y}\left(F_{z_y} h\right)\right] dxdy - \iint_R \left[\frac{\partial}{\partial x}\left(F_{z_x} 1 + \frac{\partial}{\partial y} F_{z_y} h\right)\right] h\, dxdy$$

$$= \iint_R F_z h \, dxdy + \int_D \left(F_{z_x} h \, dy - F_{z_y} h \, dx\right) - \iint_R \left[\frac{\partial}{\partial x}\left(F_{z_x}\right) + \frac{\partial}{\partial y}\left(F_{z_y}\right)\right] h \, dxdy,$$

using Green's theorem

The second integral on R.H.S. is zero because h(x, y) vanishes on the boundary D and hence, we obtain,

$$\delta J = \iint_R \left[F_z - \frac{\partial}{\partial x}\left(F_{z_x}\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right)\right] h(x, y) \, dx \, dy$$

Thus, the condition for extremum, $\delta J = 0$ implies that

$$\iint_R \left[F_z - \frac{\partial}{\partial x}\left(F_{z_x}\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right)\right] h(x, y) \, dx \, dy = 0$$

Hence by lemma (5), we have,

$$F_z - \frac{\partial}{\partial x}\left(F_{z_x}\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right) = 0$$

which is the required condition. This equation is known as Euler's equation and is a second order partial differential equation in general.

4.5.3. Example. Derive Euler's equation for the functional

$$J[z] = \iint_R \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 dx \, dy$$

Solution. Here, $F(x, y, z, z_x, z_y) = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(z_x\right)^2 - \left(z_y\right)^2$. Therefore

$$F_z = 0, \; F_{z_x} = 2z_x, \; F_{z_y} = -2z_y$$

Now, the Euler's equation is,

$$F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} = 0 \quad \Rightarrow \quad 0 - \frac{\partial}{\partial x}(2z_x) + \frac{\partial}{\partial y}(2z_y) = 0$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} = 0$$

which is the required Euler's equation. The solution of this second order partial differential equation will give the extremal of the given functional.

4.5.4. Example. Find the surface of least area spanned by a given contour.

Solution. In this case, we have to find the minimum of the functional.

$$J[z] = \iint_R \sqrt{1 + z_x^2 + z_y^2} \, dxdy$$

$\Rightarrow$ $F(x, y, z, z_x, z_y) = \sqrt{1 + z_x^2 + z_y^2}$

Therefore, $F_z = 0$; $F_{z_x} = \dfrac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}$; $F_{z_y} = \dfrac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}$

Now, the Euler's equation is

$$F_z - \frac{\partial}{\partial x}\left(F_{z_x}\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0 \qquad (1)$$

Let us calculate,

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) = \frac{\left(1+p^2+q^2\right)r - p^2r - pqs}{\left(1+p^2+q^2\right)^{3/2}}$$

where $z_x = p$; $z_y = q$; $z_{xx} = r$; $z_{xy} = z_{yx} = s$; $z_{yy} = t$

Similarly

$$\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = \frac{\left(1+p^2+q^2\right)t - q^2t - pqs}{\left(1+p^2+q^2\right)^{3/2}}$$

Using these in (1), we have,

$$\left(1+p^2+q^2\right) r - p^2r - pqs + \left(1+p^2+q^2\right) t - q^2t - pqs = 0$$

$$\Rightarrow \quad r(1+q^2) - 2pqs + t\,(1+p^2) = 0$$

The solution of this differential equation will provide the solution.

4.5.5. Example. Show that the functional $\int_0^1 \left[2x + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt$ with $x(0) = 1$ $y(0) = 1$, $x(1) = 1.5$ $y(1) = 1$ is stationary for $x = \dfrac{2+t^2}{2}$, $y = 1$.

Solution. The given functional is $\int_0^1 (2x + y'^2 + x'^2)\, dx$

Let $F = 2x + x'^2 + y'^2$

The Euler's equations are $\quad F_x - \dfrac{d}{dt} F_{x'} = 0 \qquad (1)$

$$F_y - \frac{d}{dt} F_{y'} = 0 \qquad (2)$$

Here $F_x = 2$ $F_y = 0$ $F_{x'} = 2x'$ $F_{y'} = 2y'$

So, (1) $\Rightarrow 2 - \frac{d}{dt}(2x') = 0 \Rightarrow \frac{d^2x}{dt^2} = 1 \Rightarrow \frac{dx}{dt} = t + C_1$

$$\Rightarrow \qquad x = \frac{t^2}{2} + C_1t + C_2 \qquad (3)$$

Also (2) $\Rightarrow 0 - \frac{d}{dt}(2y') = 0 \Rightarrow \frac{dy'}{dt} = 0 \Rightarrow y' = C_3$

$$\Rightarrow \qquad y = C_3t + C_4 \qquad (4)$$

So, (3) and (4) are equations of extremals

The boundary conditions are $x(0) = 1$ $y(0) = 1$ $x(1) = 1.5$ $y(1) = 1$

$x(0) = 1 \Rightarrow 0 + C_1(0) + C_2 = 1 \Rightarrow C_2 = 1$

$y(0) = 1 \Rightarrow C_3(0) + C_4 = 1 \Rightarrow c_4 = 1$

$x(1) = 1.5 \Rightarrow \frac{1}{2} + C_1(1) + C_2 = 1.5 \Rightarrow \frac{1}{2} + C_1 + 1 = 1.5 \Rightarrow C_3 = 0$

So, $C_1 = 0$, $C_2 = 1$, $C_3 = 0$, $C_4 = 1$

So, equation (3) $\Rightarrow x = \frac{t^2}{2} + 0.t + 1$, that is, $x = \frac{2+t^2}{2}$

Equation (4) $\Rightarrow y = 0.t + 1$, that is, $y = 1$

Therefore, stationary functions are $x = \frac{2+t^2}{2}$, $y = 1$.

4.6. Variable End Point Problem. So far, we have discussed the functionals with fixed end points. Sometimes it may happen that the end points lie on two given curves

$$y = \phi(x) \text{ and } y = \Psi(x).$$

Such problems are called variable end point problems. We discuss only a particular case in the form of following problem.

4.6.1. Problem. Among all curves whose end points lie on two given vertical lines x = a and x = b. Find the curve for which the functional

$$J[y] = \int_a^b F(x, y, y')\,dx \qquad (1)$$

has an extremum.

Solution. As before, we calculate

$$\Delta J = J[y+h] - J[y] = \int_a^b [F(x, y+h, y'+h') - (x, y, y')]dx$$

Using Taylor's theorem, we obtain

$$\Delta J = \int_a^b (F_y h + F_{y'} h')\, dx + \ldots$$

Then the variation δJ of the functional J[y] is given by principal linear part of ΔJ that is,

$$\delta J = \int_a^b (F_y h + F_{y'} h')\, dx$$

Here, unlike the fixed end point problem, h(x) need no longer vanish at the points a and b, so that integrating by parts the second term, we get

$$\delta J = \int_a^b \left(F_y - \frac{d}{dx} F_{y'}\right) h(x)\, dx + \left[F_{y'} h(x)\right]_{x=a}^{x=b}$$

$$= \int_a^b \left(F_y - \frac{d}{dx} F_{y'}\right) h(x)\, d(x) + F_{y'}\big|_{x=b} h(b) - F_{y'}\big|_{x=a} h(a) \qquad (2)$$

We first consider functions h(x) such that h(a) = h(b) = 0. Then, as in simplest variational problem, the condition $\delta J = 0$ implies that

$$F_y - \frac{d}{dx} F_{y'} = 0 \qquad (3)$$

Thus, in order for the curve y = y(x) to be a solution of the variable end point problem, y must be an extremal that is, a solution of the Euler's equation. But if y is an extremal, the integral in the expression (2) for δJ vanishes and then the condition $\delta J = 0$ takes the form,

$$F_{y'}\big|_{x=b} h(b) - F_{y'}\big|_{x=a} h(a) = 0$$

But since h(x) is arbitrary, it follows that

$$F_{y'}\big|_{x=a} = 0 \text{ and } F_{y'}\big|_{x=b} = 0 \qquad (4)$$

Thus, to solve the variable end point problem we must first find a general integral of Euler's equation (3) and then use the condition (4) to determine the values of arbitrary constants.

Remark.

1. The conditions (4) are some times called the natural boundary conditions.

2. Besides the case of fixed end points and the case of variable end points, we can also consider the mixed case, where one end is fixed and the other is variable.

For example, suppose we are looking for an extemum of the functional (1) w.r.t. the class of curves joining a given point A (with abscissa a) and an arbitrary point of the line x = b. In this case, the conditions (4) reduce to the single condition $F_{y'}\big|_{x=b} = 0$ and y (a) = A serves as the second boundary condition.

4.6.2. Example. Starting from the point P (a, A), a heavy particle slides down a curve in a vertical plane. Find the curve such that the particle reaches the vertical line x = b ($\neq$ a) in the shortest time.

Solution. For simplicity, we assume the point P to be origin. Then velocity of the particle,

$$v = \frac{ds}{dt} = \sqrt{1+y'^2}\,\frac{dx}{dt} \Rightarrow dt = \frac{\sqrt{1+y'^2}}{v}\,dx$$

Also, we have, $v^2 - u^2 = 2gh \Rightarrow v^2 - 0 = 2gy \Rightarrow v = \sqrt{2gy}$

So that $dt = \dfrac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$ or the total time, $T = \displaystyle\int \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$

We have to find least value of T. In this case, $F = \sqrt{\dfrac{1+y'^2}{2gy}}$

Since x is absent, so Euler's equation is taken as $F - y'F_{y'}$ = constant = C (say)

$$\Rightarrow \quad \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - y'\,.\,\frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}} = C \Rightarrow \frac{1+y'^2-y'^2}{\sqrt{1+y'^2}\sqrt{2gy}} = C \Rightarrow C^2\left(1+y'^2\right) = \frac{1}{2gy}$$

$$\Rightarrow \quad y'^2 = \frac{K}{y} - 1 \text{ where } \frac{1}{2gc^2} = K \Rightarrow \frac{dy}{dx} = \sqrt{\frac{K-y}{y}} \text{ or } \sqrt{\frac{y}{K-y}}\,dy = dx$$

Integrating, we get, $\displaystyle\int \frac{\sqrt{y}}{\sqrt{K-y}}\,dy = \int dx + C_1$ (1)

Let $y = K\sin^2\theta/2 \Rightarrow dy = K\sin\theta/2\,\cos\theta/2\,d\theta$

so that (1) becomes

$$\int \frac{\sqrt{K}\sin\theta/2}{\sqrt{K}\cos\theta/2}.\,K\sin\theta/2\,\cos\theta/2\,d\theta = x + C_1$$

$$\int K\sin^2\theta/2\,d\theta = x + C_1 \Rightarrow K\int \frac{1-\cos\theta}{2}\,d\theta\;\; x + C_1 \Rightarrow \frac{K}{2}(\theta - \sin\theta) = x + C_1$$

Or $\quad x = \dfrac{K}{2}(\theta - \sin\theta) - C_1$

Again since y = K $\sin^2\theta/2 \Rightarrow y = \frac{K}{2}(1-\cos\theta)$

Thus the solution is, $x = \frac{K}{2}(\theta - \sin\theta) - C_1$; $y = \frac{K}{2}(1-\cos\theta)$

Since the curve pass through the origin so $C_1 = 0$

So that the curve is $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$, $r = \frac{K}{2}$

This is an equation of a cycloid and value of r is determined by the second condition, namely

$$F_{y'}\big|_{x=b} = 0 \quad \Rightarrow \quad \frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}} = 0 \text{ for } x = b$$

$$\Rightarrow y' = 0 \text{ for } x = b \Rightarrow \frac{dy}{dx} = 0 \text{ for } x = b$$

Now $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = \pi$ for x = b

Now $x = r(\theta - \sin\theta) \Rightarrow r = \frac{x}{\theta - \sin\theta} = \frac{b}{\pi - \sin\pi} = \frac{b}{\pi}$

Hence the required curve finally comes as

$$x = \frac{b}{\pi}(\theta - \sin\theta)\,;\; y = \frac{b}{\pi}(1-\cos\theta)$$

4.7. Variational Derivatives. We introduce the variational (or functional) derivative, which plays the same role for functionals as the concept of the partial derivative plays for functions of n variables. We shall follow the approach to first go from the variational problem to an n-dimensional problem and then pass to the limit $n \to \infty$.

Remark. From elementary analysis we know that a necessary condition for a function of n variables to have an extremum is that all its partial derivatives vanish. Now we derive the corresponding condition for funcionals.

Consider the functional,

$$J[y] = \int_a^b F(x, y, y')\,dx \qquad (1)$$

$$y(a) = A,\; y(b) = B$$

Divide the interval [a, b into n + 1 equal subintervals by introducing the points

$$A = x_0, x_1, x_2, \ldots, x_n, x_{n+1} = b$$

and we replace the smooth function y(x) by the polygonal line with vertices

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1}) \text{ or } (a, A), (x_1, y_1), \ldots, (x_n, y_n), (b, B)$$

where $y_i = y(x_i)$. We shall denote $x_{i+1} - x_i$ by Δx. Then we approximate (1) by the sum

$$J[y_1, y_2, \ldots, y_n] = \sum_{i=0}^{n} F\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right)\Delta x \tag{2}$$

which is a function of n variables.

Now, we calculate the partial derivatives $\dfrac{\partial J(y_1, y_2, \ldots, y_n)}{\partial y_k}$

We observe that each variable y_k in (2) appears in just two terms corresponding to $i = k - 1$ and $i = k$ and these two terms are,

$$F\left(x_{k-1}, y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}\right)\Delta x + F\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right)\Delta x$$

Thus we have,

$$\frac{\partial J}{\partial y_k} = F_y\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right)\Delta x + F_{y'}\left(x_{k-1}, y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}\right) - F_{y'}\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right)\Delta x \tag{3}$$

Dividing (3) by Δx, we get

$$\frac{\partial J}{\partial y_k \Delta x} = F_y\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right) - \frac{1}{\Delta x}\left[F_{y'}\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right) - F_{y'}\left(x_{k-1}, y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}\right)\right] \tag{4}$$

The expression $\partial y_k \, \Delta x$ appearing in denominator on the left has a direct geometric meaning.

As $\Delta x \to 0$, the expression (4) gives

$$\frac{\delta J}{\delta y} \equiv F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') \qquad \ldots\ldots\ldots(*)$$

This is known as the variational derivative of the functional.

Remark. By equation (*), we observe that variational derivative $\dfrac{\delta J}{\delta y}$ is just the left – hand side of Euler's equation and for extremal variational derivative of the functional under consideration should vanish at every point. This is the analog of the situation encountered in elementary analysis, where a necessary condition for a function of n variables to have an extremum is that all in partial derivatives vanish.

4.7.1. Another definition of variational derivative. Let J [y] be a functional depending on the function y(x) and suppose we give y(x) and increment h(x) which is non zero only in the neighbourhood of a point x_0. Let Δ_σ be the area lying between the curve y = y(x) and y = y(x) + h(x).

Now, dividing the increment J(y + h) − J [y] of the functional J [y] by the area Δ_σ we obtain the ratio,

$$\frac{J[y+h] - J[y]}{\Delta_\sigma} \qquad (1)$$

Now, we let the area Δ_σ tend to zero in such a way that both max. $|h(x)|$ and the length of the interval in which h(x) is non –zero tend to zero. Then, if the ratio (1) converges to a limit as $\Delta\sigma \to 0$, this limit is called the variational derivative of the functional J[y] at the point x_0 and is denoted by $\left.\frac{\delta J}{\delta y}\right|_{x = x_0}$.

Remark.

1. It is clear from the definition of the variational derivative that the increment

$$\Delta J \equiv J[y+h] - J[y] = \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_0} + \in \right\} \Delta\sigma$$

where $\in \to 0$ as both max. $|h(x)|$ and the length of the interval in which h(x) is non vanishing tend to zero.

2. It also follows that in terms of variational derivative, the differential or variation of the functional J[y] at the point x_0 is given by

$$\delta J = \left.\frac{\delta J}{\delta y}\right|_{x=x_0} \Delta\sigma$$

4.7.2. Invariance of Euler's Equation. In this section we show that whether or not a curve is an extremal is a property which is independent of the choice of the coordinate system.

For this, consider the functional $J[y] = \int_a^b F(x, y, y')\,dx$ (1)

Now we introduce another system of coordinates by substituting,

x = x (u, v) and y = y (u, v) such that the Jacobian $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$ (2)

Then the curve given by the equation y = y(x) in the xy, plane corresponds to the curve given by some equation

v = v(u) in the uv – plane.

Now, we have

$$\frac{dx}{du} = \frac{\partial x}{\partial u}\frac{du}{du} + \frac{\partial x}{\partial v}.\frac{dv}{du} = x_u + x_v$$

$$\frac{dy}{du} = \frac{\partial y}{\partial u}\frac{du}{du} + \frac{\partial y}{\partial v}.\frac{dv}{du} = y_u + y_v\, v'$$

which gives

$$\frac{dy}{dx} = \frac{y_u + y_v\ v'}{x_u + x_v\ v'} \text{ ad } dx = (x_u + x_v\ v')du$$

By these substitutions, functional (1) transforms into the functional,

$$J_1[v] = \int_{a_1}^{b_1} F\left[x(u,v), y(u,v), \frac{y_u + y_v\ v'}{x_u + x_v\ v'}\right](x_u + x_v\ v')du = \int_{a_1}^{b_1} F_1(u, v, v')du \text{ (say)} \qquad (3)$$

We now show that if y = y(x) satisfies the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad (4)$$

corresponding to the original functional J[y], then v = v(u) satisfies the Euler's equation.

$$\frac{\partial F_1}{\partial v} - \frac{d}{du}\left(\frac{\partial F_1}{\partial v'}\right) = 0 \qquad (5)$$

corresponding to the new functional $J_1[v]$. To prove this, we use the concept of variational derivative.

Let $\Delta\sigma$ denotes the area bounded by the curves y = y(x) and y = y(x) + h(x) and let $\Delta\sigma_1$ denotes the area bounded by corresponding curves v = v(u) and v = v(u) + η(u) in the uv – plane.

Now (by a standard result for the transformation of areas) as $\Delta\sigma$ and $\Delta\sigma_1$ tend to zero, the ratio $\Delta\sigma/\Delta\sigma_1$ approaches the Jacobian $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ which is non zero by (2).

Thus if $\lim\limits_{\Delta\sigma\to 0} \dfrac{J[y+h] - J[y]}{\Delta\sigma} = 0$, then, $\lim\limits_{\Delta\sigma\to 0} \dfrac{J[v+\eta] - J_1[v]}{\Delta\sigma_1} = 0$

It follows that if y(x) satisfies (4), then v(x) satisfies (5). This proves invariance of Euler's equation on changing the coordinate system.

Remark. In solving Euler's equation sometimes change of variables can be used for simplicity. Because of the invariance property, the change of variables can be made directly in the integral rather than in Euler;s equation and we can then write Euler's equation for new integral.

4.7.3. Example. Find the extemals of the functional $\int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta$ where r = r(θ)

Solution. Let J [r (θ)] = $\int_{\theta_1}^{\theta_2} \sqrt{(r^2 + r'^2)}\, d\theta$

Put x = r cosθ, y = r sinθ $\Rightarrow$ $\dfrac{dx}{d\theta} = -\, r\sin\theta + \dfrac{dr}{d\theta}\cos\theta$

and $\dfrac{dy}{d\theta} = r\cos\theta + \dfrac{dr}{d\theta}\sin\theta$

Squaring and adding, we get,

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2 + r'^2$$

Thus, we have

$$\int \sqrt{r^2 + r'^2}\, d\theta = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int \sqrt{(dx)^2 + (dy)^2}$$

$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + y'^2}\, dx$$

Suppose at $\theta = \theta_1$, $x = x_1$ and at $\theta = \theta_2$, $x = x_2$ so that the given functional becomes,

$$J[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

which has been solved earlier and gives the solution of the type

$$y = mx + c$$

Thus, the extremals are $r \sin\theta = m\, r\, x\!\cos\theta + c$

4.7.4. Exercise. Find the extremal of the functional $\int_{\theta_1}^{\theta_2} r \sin\theta \sqrt{r^2 + r'^2}\, d\theta$ using the transformation $x = r\cos\theta$, $y = r\sin\theta$.

4.8. The fixed end point problem for n unknown functions.

4.8.1. Theorem. A necessary condition for the curve $y_i = y_i(x)$ $(i = 1, 2, \ldots, n)$ to be an extremal of the functional

$$J[y_1, y_2, \ldots, y_n] = \int_a^b F(x_1, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx$$

is that the functions $y_i(x)$ satisfy the Euler's equations.

$$F_{y_i} - \frac{d}{dx}\left(F_{y_i'}\right) = 0 \qquad (i = 1, 2, \ldots, n)$$

Proof : Let $F(x_1, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')$ be a function with continuous first and second derivatives w.r.t. all its arguments. Consider the functional.

$$J[y_1, y_2, \ldots, y_n] = \int_a^b F(x_1, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx \qquad (1)$$

satisfying the boundary conditions,

$$y_i(a) = A_i,\ y_i(b) = B_i\ (i = 1, 2, \ldots, n) \qquad (2)$$

We replace each $y_i(x)$ by a varied function $y_i(x) + h_i(x)$ where both $y_i(x)$ and $y_i(x) + h_i(x)$ satisfy the boundary conditions (2). For this, we must have,

$$h_i(a) = h_i(b) = 0 \qquad (i = 1, 2, \ldots, n)$$

We now calculate the increment

$$\Delta J = J[y_1 + h_1, \ldots, y_n + h_n] - J[y_1, \ldots, y_n]$$

$$\Rightarrow \Delta J = \int_a^b F(x_1, y_1 + h_1, \ldots, y_n + h_n + y_1' + h_1', \ldots, y_n' + h_n') - F(x_1, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx$$

Using Taylor's theorem, $\Delta J = \int_a^b \sum_{i=1}^{n} \left(F_{y_i} h_i + F_{y_i'} h_i'\right) dx + \ldots$

The integral on R.H.S. represents the principal linear part of ΔJ and hence the variation of $J[y_1, \ldots, y_n]$ is

$$\delta J = \int_a^b \sum_{i=1}^{n} \left(F_{y_i} h_i + F_{y_i'} h_i'\right) dx$$

Since all the increments $h_i(x)$ are independent, we can choose one of them arbitrarily setting all others equal to zero, so that the necessary condition $\delta J = 0$ for an extremum implies

$$\int_a^b \left(F_{y_i} h_i + F_{y_i'} h_i'\right) dx = 0 \qquad (i = 1, 2, \ldots, n)$$

Using lemma (4) (earlier proved), we obtain

$$F_{y_i} = \frac{d}{dx}\left(F_{y_i'}\right) \quad \text{or} \quad F_{y_i} - \frac{d}{dx} F_{y'_i} = 0 \qquad (i = 1, 2, \ldots, n)$$

which are required Euler's equations. This is a system of n second order differential equations, its general solution contains 2n arbitrary constants, which are determined from the boundary conditions (2).

4.8.2. Example. Find the extremals of the functional $J[y, z] = \int_{x_0}^{x_1} \left(2yz - 2y^2 + y'^2 - z'^2\right) dx$

Solution : Euler's equations are $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad (1)$

and $\qquad \frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = 0 \qquad (2)$

Here $f = 2yz - 2y^2 + y'^2 - z'^2$

which gives $\frac{\partial f}{\partial y} = 2z - 4y\ ;\ \frac{\partial f}{\partial y'} = 2y'\ ;\ \frac{\partial f}{\partial z} = 2y\ ;\ \frac{\partial f}{\partial z'} = -2z'$

Thus equation (1) and (2) reduce to,

$$2z-4y-\frac{d}{dx}\left(2y'\right)=0 \Rightarrow z-2y-\frac{d^2y}{dx^2}=0 \tag{3}$$

and $$2y-\frac{d}{dx}\left(-2z'\right)=0 \Rightarrow y+\frac{d^2z}{dx^2}=0 \tag{4}$$

From (3), we have $z = 2y + \frac{d^2y}{dx^2} \Rightarrow \frac{d^2z}{dx^2} \equiv 2\frac{d^2y}{dx^2}+\frac{d^4y}{dx^4}$

Putting in (4), we get, $y + 2\frac{d^2y}{dx^2}+\frac{d^4y}{dx^4}=0 \Rightarrow (D^4+2D^2+1)\,y=0$

Aux. Equation is $m^4+2m^2+1=0 \Rightarrow (m^2+1)^2=0 \Rightarrow m=\pm i, \pm i$

Hence, the solution is

$$y = (Ax+B)\cos x + (Cx+D)\sin x \tag{5}$$

and z can be obtained by using the relation, $z = 2y + \frac{d^2y}{dx^2}$

which comes out to be, $z = (Ax+B)\cos x + (Cx+D)\sin x + 2C\cos x - 2A\sin x$ (6)

Equations (5) and (6) are required solutions where A, B, C, D can be determined by the boundary conditions.

4.8.3. Exercise. Find the extremals of the functionals

$$J[x_1, x_2] = \int_0^{\pi/2}\left(x_1'^2+x_2'^2+2x_1x_2\right)dt \text{ subject to boundary conditions}$$

$$x_1(0)=0,\ x_1(\pi/2)=1,\ x_2(0)=0,\ x_2(\pi/2)=-1$$

Answer. $x_1(t)=\sin t$, $x_2(t)=-\sin t$

4.9. Check Your Progress.

1. Find the extremals of the functional $\int_0^{\pi}\left[y'^2-y^2+uy\cos x\right]dx$, given that $y(0) = 0 = y(1)$.

Answer. $y = (B + x)\sin x$ where B is an arbitrary constant.

4.10. Summary. In this chapter, we observed that to find maxima and minima of functional small changes in functions and functionals were made to derive the required equations and hence some ordinary / partial differential equations were obtained, solving which we achieved the functions which result the functional in extreme values.

Books Suggested:

1. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.
2. Gelfand, J.M., Fomin, S.V., Calculus of Variations, Prentice Hall, New Jersey, 1963.